Semigroup Approach to Representation Theory of Infinite Wreath Products

A Thesis

Submitted to the Faculty

of

Drexel University

by

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in partial fulfillment of the

requirements for the degree

of

Doctor of Philosophy

August 2010

DEDICATIONS

To my parents, Hak Soon Yoo and In Pyong Yoo

My wife, Dr. Seunglee Kwon

My colleague, Dr. Arkady Kitover

With all my love and gratitude

ACKNOWLEDGMENTS

I want to thank my Ph.D. advisor Dr. Robert Boyer who has been a dedicated professor and a devoted mentor during my Ph.D. life. It has been an honor to be one of his Ph.D. students. Dr. Boyer has given me moral support, has shared his insights and wisdom in mathematics, and encourages me to pursue Ph.D. at Drexel University.

I would like to thank Dr. Robert Boyer again for introducing me to this problem and for attention to my work. Dr. Grigori Olshanski kindly attracted my attention to his article [26] and suggested that the technique developed there can be useful, after an appropriate modification, to the subject of my research.

I would like to thank my Ph.D. committee: Dr. R. Andrew Hicks (also Graduate Advisor), Dr. Justin Smith, Dr. Arkady Kitover, and Dr. Dmitry Kaliuzhnyi-Verbovetskyi. I also thank Dr. Hugo Woerdeman, Dr. Shari Moskow, Dr. Ronald Perline, Ms. Patricia Henry-Russell, Ms. Malinda Gilchrist, and Mr. Byron Greene for caring and support from the Mathematics Department. Thanks to my best friends, Jason Reed and Kyounghwa Bae who have always supported me.

I would like to thank my parents, my parents in-law (Soonhee Son and Heungjoo Kwon), my sister (Si Ah Yoo), and my wife. Without their help and support, I would not have finished my Ph.D. at Drexel University. Last but not least I want to thank the Community College of Philadelphia where I received my initial mathematics education and where I work as an Assistant Professor at the present.

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ABSTRACT

Semigroup Approach to Representation Theory of Infinite Wreath Products
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We consider the group $S(\infty)$ of all permutations of the set of naturals, $\mathbb{N}$, with topology of pointwise convergence. Lieberman proved that any representation T of the $S(\infty)$ is a direct sum of irreducible representations; in particular, T generates a von Neumann algebra of type I. Lieberman's proof is very complicated and can hardly be applied to more general semigroups. Olshanski found another proof of Lieberman's theorem based on semigroup approach. Using this approach we extend the Lieberman's Theorem on the case of the wreath product of $S(\infty)$ with Z_2 endowed with the topology of pointwise convergence.

1. INTRODUCTION

The representation theory of non-locally compact groups is a very rich, difficult and interesting branch of mathematics. Especially for big groups[1] the representation theory is useful for natural problems of classical representation theory and physical applications. We introduce the group $S^0(\infty)$ as $\bigcup\limits_{n=1}^{\infty} S(n)$ where $S(n)$ is the group of all permutations on the set $\{1, 2, \cdots, n\}$. We consider $S^0(\infty)$ with the topology of pointwise convergence; the subgroups have a fundamental system of neighborhoods of the identity in this topology. The completion of $S^0(\infty)$ in pointwise convergence topology[2] is the infinite symmetric group $S(\infty)$ of all permutations on the set of natural numbers, $\mathbb{N}$. So $S^0(\infty)$ is a countable discrete group but $S(\infty)$ is uncountable. Notice that $S^0(\infty)$ is isomorphic to a subgroup of all permutation matrices on ℓ_2 and therefore of can be considered as a subgroup of $U(\ell_2)$. Then $S(\infty)$ can be identified with the closure of $S^0(\infty)$ in strong operator topology. The infinite symmetric group $S(\infty)$ has the topology in which the subgroups form a fundamental system of neighborhoods of the identity. The topological groups $S^0(\infty)$ and $S(\infty)$ are non-locally compact. Since the infinite symmetric groups and their representations are used in the fields of modern theoretical physics such as gauge theories, gravitation, and integrable systems, they occupy a special place in the theory of representations and some unexpected phenomena occur. One of these phenomena is the interaction of infinite-dimensional groups with semigroups. One new approach is that the representation of an infinite-dimensional group can be embedded into that one of a semigroup.

The group $S^0(\infty)$ with the discrete topology is an example of a non-type I group for which a regular representation can not be decomposed into irreducible representations uniquely. The group $S^0(\infty)$ endowed with the discrete topology was studied in [21] as an example

[1]this terminology was first time used by A. Vershik, big groups mean in this thesis the infinite-dimensional groups.

[2]$\sigma_m \to \sigma$ if $\forall n \in \mathbb{N},\ \exists m(n)$ such that if $m > m(n)$ then $\sigma_m(i) = \sigma(i),\ i = 1, 2, \cdots, n$.

of ICC (infinite conjugacy classes)-group, where it was shown that any regular representation of an ICC-group is a II_1 factor representation. (So $S^0(\infty)$ with discrete topology is non-type I). Factor representations of type II have been studied also in [38], [39], and [40]. The infinite symmetric group $S(\infty)$ endowed with the pointwise convergence topology was studied by Lieberman. The countable families of irreducible unitary representations of $S(\infty)$ are discussed in [19] and the uncountable case in [22] and [23]. Later, Olshanski developed in [26] a powerful technique which permits us to obtain a classification of all the unitary representations of $S(\infty)$ using a semigroup method.

Recently, many researchers became interested in wreath products of groups since they provide nice examples of non-locally compact groups. Since the wreath product is not commutative, we have to distinguish the wreath product of A with B from the wreath product of B with A. Another approach to this wreath product is discussed in [4], [10], [12] and is based on Young diagrams, graphs, and the study of finite (or infinite) characters.

In this thesis, we discuss the wreath product of the infinite symmetric group $S(\infty)$ with Z_2 where Z_2 is the standard group[3], $Z_2 = \{0, 1\}$ with the operation $0 + 0 = 1,\ 0 + 1 = 1 + 0 = 1,\ 1 + 1 = 0$. The main goal of this thesis is to give a proof that any representation of the wreath product of the infinite symmetric group $S(\infty)$ with Z_2 generates a von Neumann algebra[4]$\mathcal{M}$ of type I. In fact, the von Neumann algebra $\mathcal{M}$ is a discrete sum of algebras $*$-isomorphic to the algebra $L(H)$ for some Hilbert space H. The proof is based on Olshanski's semigroup method. In particular, any continuous unitary representation of the wreath product of $S(\infty)$ with Z_2 can be decomposed into a direct sum of irreducible representations.

The outline of the thesis is as follows. In Chapter 2, we introduce the infinite symmetric group $S(\infty)$ and representations of semigroups. Then we provide tame representations for $S^0(\infty)$ and we state Lieberman Theorem for $S(\infty)$. In Chapter 2.4 and 2.5 we consider

[3]Our method can be easily extended to the wreath product of $S(\infty)$ with Z_n, where $n \in \mathbb{N}$.

[4]The definitions related to von Neumann algebras and their types are collected (see Appendix H).

the finite symmetric group $S(n)$ using a Young diagram. We construct all irreducible representations of the finite symmetric group $S(n)$ using Specht modules. In Chapter 3, we define the wreath product of the infinite symmetric group $S(\infty)$ with Z_2 and we show the relation between the infinite symmetric group $S(\infty)$ and some groups of matrices with the wreath product of infinite symmetric group $S(\infty)$ with Z_2. Then we describe the semigroups generated by the wreath product of the infinite symmetric group $S(\infty)$ with Z_2 and the subgroups as groups of permutations. In Chapter 4, we describe the unitary representations and tame representations of infinite wreath products. We prove an extension of Lieberman's Theorem for the wreath product of infinite symmetric group $S(\infty)$ with Z_2 in Chapter 5.

In the future, it would be of interest to prove that the representation of the wreath product of infinite symmetric group $S(\infty)$ endowed with the topology of pointwise convergence with G where G is a compact (locally compact) group is of type I. My thesis can be considered as a initial step toward the solution of this problem.

2. THE INFINITE SYMMETRIC GROUP $S(\infty)$ AND REPRESENTATIONS OF SEMIGROUPS

2.1 The infinite symmetric group $S(\infty)$ and its subgroups

In this and the next sections we follow in the steps of Olshanski [26]. $S(\infty)$ denotes the group of all the permutations of the set $\mathbb{N}$ (i.e. $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$). So $S(\infty)$ is uncountable. The group $S(\infty)$ acts on $\mathbb{N}$ in an obvious way. Its action can be written as $\sigma(i) = j$ where $\sigma \in S(\infty),\ i, j \in \mathbb{N}$. For example, if

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 1 & 4 & 3 & 5 & 6 & \cdots \end{bmatrix}.$$

then $\sigma(3) = 4$.

A permutation $\sigma \in S(\infty)$ is called finite if $Card\{i \in \mathbb{N} \mid \sigma(i) \neq i\}$ is finite. We will denote by $S^0(\infty)$ the set of all the finite permutations. $S^0(\infty)$ is a countable normal subgroup in $S(\infty)$.

We will now describe three important subgroups of $S(\infty)$: $S_n(\infty),\ \tilde{S}_n(\infty),\ S(n)$;

$$S_n(\infty) = \{\sigma \in S(\infty) \mid \sigma(1) = 1, \sigma(2) = 2, \cdots, \sigma(n) = n\}.$$

For example, if

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & 2 & 3 & 5 & 4 & 7 & \cdots \end{bmatrix}.$$

then $\sigma \in S_3(\infty)$.

$$\tilde{S}_n(\infty) = \{\sigma \in S(\infty) \mid \sigma\{1, \cdots, n\} = \{1, \cdots, n\}\}.$$

For example, if

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 3 & 1 & 2 & 5 & 4 & 7 & \cdots \end{bmatrix}.$$

then $\sigma \in \tilde{S}_3(\infty)$. Obviously $S_n(\infty) \subset \tilde{S}_n(\infty)$.

The group of all permutations of $\{1, 2, \cdots, n\}$ is denoted by $S(n)$.

$$S(n) = \{\sigma \in S(\infty) \mid \sigma(i) = i, \ \text{ for all } i \geq n+1\}.$$

For example, if

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 3 & 1 & 2 & 4 & 5 & 6 & \cdots \end{bmatrix}.$$

then $\sigma \in S(3)$.

We state the concept of a topological group following [31]. We shall give here the definition of a topological group and some examples.

Definition 2.1. *A set G of elements is called a topological group if*

1. *G is a group.*

2. *G is a topological space.*

3. *The group operations in G are continuous in the topological space G. In other words, if a and b are two elements of the set G, then for every neighborhood W of the element ab^{-1} there exist neighborhoods U and V of the elements a and b such that $UV^{-1} \subset W$.*

The examples of topological groups are

1. The additive group of real numbers with usual topology,

2. $GL(d, \mathbb{C})$ with topology $dist(g, h) = \sqrt{\sum_{i,j=1}^{d} |g_{ij} - h_{ij}|}$,

3. The subgroups of $GL(d, \mathbb{C})$; $SL(d, \mathbb{C}),\ SO(d),\ U(d),\ SU(d)$,

4. All Lie groups,

5. 1-dimensional torus group,

6. The infinite symmetric group with discrete topology.

Now we state definition of the fundamental system of neighborhoods of the identity.

Definition 2.2. *A system $\{U_\alpha\}$ is called a fundamental system of open neighborhoods of the identity if for any neighborhood U of the identity there is $U_\alpha \in \{U_\alpha\}$ such that $U_\alpha \subseteq U$.*

The subgroups $S_n(\infty)$ and $\tilde{S}_n(\infty)$ are open in $S(\infty)$. The subgroups $S_n(\infty)$ form a fundamental system of neighborhoods of the identity in the pointwise convergence topology on $S(\infty)$ since for given $\sigma \in S(\infty)$ and a number n, we have $S_m(\infty) \subseteq \sigma^{-1} S_n(\infty)\sigma$ when m is very large that $m \geq \sigma(1), \cdots, m \geq \sigma(n)$. Notice that it is more convenient to us the topology of pointwise convergence directly.

Proposition 2.1. *The infinite symmetric group $S(\infty)$ with the pointwise convergence topology is a non-locally compact topological space.*

Proof: We will prove the statement by contradiction. Let e be the identity element. If $S(\infty)$ is locally compact, then for some n the neighborhood V_n of e would be compact where $V_n = \{\tau \in S(\infty) : \tau(i) = i, i = 1, 2, \cdots, n\}$. Because $S(\infty)$ is metrizable, the compactness of V_n would mean that if we have a sequence $\tau_m \in V_m$ then we can find a subsequence τ_{m_k} such that $\tau_{m_k} \to \sigma_1 \in S(\infty)$. Now let $\tau_m(i) = i,\ i = 1, \cdots, n$ and $\tau_m(n+1) = mn$, then $\tau_m(n+1) \to \infty$ as $m \to \infty$ and no subsequence of τ_m can converge. This is contradiction. Notice that if we take finite τ_m then we see that $S^0(\infty)$ is non-locally compact. $\Diamond$

Notice that

1. $\tilde{S}_n(\infty)$ is the semidirect product of its normal subgroup $S_n(\infty)$ and $S(n)$.

2. $S_n(\infty) \cap S(n) = \{e\}$.

3. $S^0(\infty) = \bigcup_{n \geq 1} S(n)$.

4. $S_n(\infty)$ and $\tilde{S}_n(\infty)$ are open in $S(\infty)$ and $S^0(\infty)$ is dense in $S(\infty)$.

To state the next result let us recall that to every bounded linear operator on ℓ_2 corresponds an infinite matrix. The product of permutations in $S(\infty)$ corresponds to the product of matrices. $S(n)$ can be considered as the group of all square 0 or 1 matrices and only one entry equal to 1 in each row and each column of order n.

Proposition 2.2. $S(\infty)$ *is isomorphic to the group of all infinite matrices whose entries are only* 0 *or* 1 *and in any row or column the numbers of* $1'$*s is exactly one. More explicitly, for any* $\sigma \in S(\infty)$*, the matrix corresponding to* σ *is as follows:*

$$\sigma_{ij} = \begin{cases} 1 & \textit{if } \sigma(i) = j, \\ 0 & \textit{otherwise} . \end{cases}$$

Proof: (See Chapter 3 and 4 in Fabec's text [6]; especially Lemma III.3 and Proposition IV. 1) Let H denote a separable complex Hilbert space. The strong operator topology on $L(H)$ has a base of neighborhoods of an operator T_0 of the form

$$V(T_0; x_1, \ldots, x_m; \epsilon) = \{T \in L(H) : \|(T - T_0)x_j\| < \epsilon,\ 1 \leq j \leq m\}$$

where $x_1, \ldots, x_m$ are in H and $\epsilon > 0$.

Now the unit ball in $L(H)$ is a complete metric space in the strong operator topology with the metric

$$d(A,B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A-B)e_n\|$$

where $\{e_n\}$ is a orthonormal basis for H.The weak and strong topologies coincide on $U(H)$.

A completely metrizable space is a topological space for which there exists at least one metric which induces the same topology and is a complete metric.

Let $S^0(\infty) = \bigcup_{n=1}^{\infty} S(n)$. We can identify $S^0(\infty)$ with a subgroup of $U(\ell_2)$ where $\sigma \in S^0(\infty)$ corresponds to a permutation operator P_σ with finite support; that is, $P_\sigma - I$ has finite rank.

Since the weak and strong operator topologies coincide on $U(\ell_2)$, let $V \in U(\ell_2)$ be in the closure of $S^0(\infty)$. Then there must be a net $P_\alpha \in S^0(\infty)$ such that $P_\alpha \to V$; on the other hand

$$(P_\alpha e_m, e_n) \to (Ve_m, e_n)$$

for all e_m, e_n. Since $(P_\alpha e_m, e_n)$ can attain only two values: 0 or 1, the same must be true for (Ve_m, e_n). Hence, V can also be viewed as a permutation operator but it may not longer have finite support; that is, $V - I$ may have infinite rank.

Finally, we indicate why any permutation operator must lie in the closure of $S^0(\infty)$. Let $\epsilon > 0$ be given. Choose n_0 so the tail $\sum_{n=n_0}^{\infty} \frac{1}{2^n} < \epsilon$. Consider the vectors $e_1, e_2, \ldots, e_{n_0}$ together with $Ve_1, \ldots, Ve_{n_0}$. Then there exists a permutation $\sigma_0 \in S^0(\infty)$ such that $V_{\sigma_{n_0}} e_j = Ve_j$ for $1 \leq j \leq n_0$. Then $d(V_{\sigma_{n_0}}, V) < \epsilon$. In particular, V must lie in the closure of $S^0(\infty)$ in $U(\ell_2)$.

We conclude that the group of all permutations on $\mathbb{Z}^+$ coincides with the closure of $S^0(\infty)$ in $U(\ell_2)$. $\Diamond$

Example 2.1. *Let*

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 1 & 4 & 3 & 6 & 5 & \cdots \end{bmatrix},$$

then the corresponding matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2.2 Semigroups with involution and unitary representations

Let H be a complex Hilbert space.

Definition 2.3. *A semigroup is a set Φ together with a binary operation satisfying two conditions:*

1. *Closure:* $\forall\ a,\ b \in \Phi, \quad ab \in \Phi$,
2. *Associativity:* $\forall\ a,\ b,\ c\ \in \Phi, \quad (ab)c = a(bc)$.

Note that a semigroup Φ does not need to have an identity element and elements of Φ do not have to have inverses. We describe now semigroups with involution. An involution $*$ is a bijection $\phi \mapsto \phi^*$ of Φ on Φ satisfying

$$\phi^{**} = \phi, \quad (\phi_1\phi_2)^* = \phi_2^*\, \phi_1^* \qquad \forall \phi, \phi_1, \phi_2 \in \Phi.$$

We assume that Φ possesses an identity element 1 and $1^* = 1$. Notice that if Φ is a group then the inverse map $*: \Phi \to \Phi$ defined by $\phi^* = \phi^{-1}$ is an involution.

Let us describe the notion of a representation of a semigroup Φ. We denote by $L(H)$ the algebra of all bounded linear operators in H. Then the involution of $L(H)$ is the usual conjugation of operators. So $\Phi(H)$ is a semigroup with involution.

A representation ρ of a semigroup Φ is any morphism

$$\rho : \Phi \to \Phi(H)$$

which commutes with involution such that $\Phi(1) = I$. If Φ is a topological semigroup then we will assume additionally that ρ is a continuous map from Φ to $L(H)$, where $L(H)$ is endowed with the strong operator topology. If $H \subset H_\rho$ is invariant under Φ then $H^\perp \subset H_\rho$ is also invariant.

A representation ρ is called irreducible if there is no proper closed ρ-invariant subspace in H_ρ (in other words, H_ρ and $\{0\}$ are the only ρ-invariant closed subspaces of H_ρ).

We define a unitary representation of a topological group.

Definition 2.4. *Let G be a topological group and H be a Hilbert space. A unitary representation of G is a continuous homomorphism*

$$\pi : G \to U(H_\pi),$$

of G into the group $U(H_\pi)$ of unitary operators on H where H_π is the the representation space of π. The dimension of H_π is called the dimension of representation π.

Definition 2.5. *Let $\pi : G \to U(H_\pi)$ be an unitary representation of G. A closed linear subspace W of H_π is called invariant under π if $\pi(g)W \subseteq W$ for all $g \in G$.*

Any 1-dimensional representation is irreducible and if G is abelian, the dimension of every irreducible representation is 1.

We now state some important lemmas which will be used in the later chapters with proofs in [26].

Lemma 2.1. *The representation ρ is irreducible if and only if the linear span of operators, $\rho(\phi),\ \phi \in \Phi$, is weakly dense in $L(H_\rho)$.*

Definition 2.6. *A family of vectors $\{\eta_\alpha\},\ \alpha \in A$ in H is called total in H if its linear span is dense in H.*

Definition 2.7. *A vector $\eta \in H_\rho$ is called cyclic if $\{\rho(\phi)\eta\},\ \eta, \phi \in \Phi$ is total in H_ρ. A subspace $S \subseteq H_\rho$ is called cyclic if $\{\rho(\phi)\eta\}, \rho \in \Phi, \eta \in S$ is total in $H_\rho,\ \eta \in S$.*

Notice that if the representation ρ is irreducible, any nonzero vector (or subspace) of H_ρ is cyclic.

Lemma 2.2. *Let $\{\eta_\alpha\}$ and $\{\eta'_\alpha\},\ \alpha \in A$ be two total families in H and H' respectively. Suppose $(\eta_\alpha, \eta_\beta) = (\eta'_\alpha, \eta'_\beta),\ \alpha, \beta \in A$. Then there exists a linear isometry $H \to H'$ which maps η_α into η'_α.*

Lemma 2.3. *A unitary representation of a compact group G can be decomposed into a direct sum of irreducible representations.*

Definition 2.8. *Let ρ and π be two unitary representations of a group G. A linear operator τ of H_ρ into H_π satisfying*

$$\tau\rho(\phi) = \pi(\phi)\tau, \text{ for all } \phi \in G,$$

is called an intertwining operator between ρ and π.

Definition 2.9. *Two unitary representations $\rho : G \to U(H_\rho)$ and $\pi : G \to U(H_\pi)$ of G are called unitary equivalent (denoted by $\rho \cong \pi$) if there exist a unitary $W : H_\rho \to H_\pi$ such that $\pi(g) = W\rho(g)W^{-1}$ for all $g \in G$. W is sometimes called an intertwining operator between ρ and π.*

We can introduce now the dual of a topological group.

Definition 2.10. *The set of equivalence classes of irreducible unitary representations of G (denoted by $\widehat{G}$) is called the dual of G. Each class consists of unitary equivalent irreducible representations.*

Lemma 2.4. *Let ρ and π be two representations of a semigroup Φ. Suppose that there exist cyclic vectors $\eta \in H_\rho$ and $\xi \in H_\pi$ such that*

$$(\rho(\phi)\eta, \eta) = (\pi(\phi)\xi, \xi), \quad \text{for every } \phi \in \Phi.$$

Then ρ and π are equivalent.

Lemma 2.5. *Let ρ and π be two finite-dimensional irreducible representations of group G and τ be an intertwining operator between ρ and π. Then either $\tau = 0$ or τ is a linear isomorphism of H_ρ onto H_π.*

Notice that if semigroup Φ has a zero element and ρ is irreducible representations of Φ, then $\rho(0) = 0$ or $dim(H_\rho) = 1$ and $\rho(\phi) = 1$ for all $\phi \in G$.

2.3 Tame representations and Lieberman Theorem for $S(\infty)$

Let T be a unitary representation of $S(\infty)$ and let H_T be the Hilbert space of the representation. We introduce H_T^n, the subspace of all $S_n(\infty)$-invariant vectors in H_T. It is clear that $H_T^n \subseteq H_T^{n+1}$. Let

$$H_T^\infty = \bigcup_{n=1}^{\infty} H_T^n.$$

This is an algebraically invariant subspace under the action of the group $S(\infty)$ in H_T. We introduce now a very important definition.

Definition 2.11. *[27] A unitary representation T^0 of the group $S^0(\infty)$ is called tame if it satisfies one of the equivalent conditions:*

(a) H_T^∞ *is dense in* $H(T)$.

(b) T^0 *can be extended to a continuous unitary representation of the topological group* $S(\infty)$.

(c) T^0 *is continuous in the pointwise convergence topology on* $S^0(\infty)$.

Definition 2.12. *[26] Let* π *be an element of* $\widehat{S(n)}$*, the finite set of all equivalence classes of representations of* $S(n)$*. Consider* π *as a representation of* $\tilde{S}_n(\infty)$ *which is trivial on* $S_n(\infty)$*.* $T(n, \pi)$ *is the unitary representation of* $S(\infty)$ *which is induced by* π*.*

We cite Lieberman Theorem for $S(\infty)$[1] as stated in [26].

Theorem 2.1. *(a) The representations* $\{T(n, \pi), n = 1, 2, \cdots, \pi \in \widehat{S(n)}\}$*, together with the trivial representation, exhaust all the continuous irreducible unitary representations of* $S(\infty)$*.*

(b) Any continuous unitary representation of $S(\infty)$ *can be decomposed into a direct sum of irreducible representations.*

Proof: It can be found in [26]. $\Diamond$

Notice that Lieberman proved in 1972 (see [19]) that any representation T of the $S(\infty)$ is a direct sum of irreducible representations; in particular, T generates a von Neumann algebra $\mathcal{M}$ of type I. Olshanski in 1985 (see [26]) found another proof of Lieberman's Theorem based on semigroup approach.

2.4 Representations of finite groups

In this section, we consider representations of finite groups. We cite without proofs from [2], [7], [18] and [34]. In this section unless we state otherwise G will be a finite group.

[1]For the original proof of Lieberman Theorem [19] (See Appendix A)

Let V be a Banach space. The general linear group of V (denoted by $GL(V)$) is the set of all invertible linear transformation of V to itself (so $GL(V)$ is the group of automorphisms of V). The representations of finite groups can be defined by using group homomorphism.

Definition 2.13. *A representation of G on V is a homomorphism*

$$\rho : G \to GL(V)$$

of G to $GL(V)$. The dimension of V (denoted by $dim(V)$) is called the dimension of ρ.

We introduce now the group algebra of a group G.

Definition 2.14. *The group algebra $\mathbb{C}[G]$ is the set all formal linear combinations of elements of G:*

$$\mathbb{C}[G] = \{\sum_{i=1}^{k} c_i g_i : \quad c_i \in \mathbb{C},\ g_i \in G\}.$$

Proposition 2.3. *$\mathbb{C}[G]$ is a complex semisimple algebra.*

If we do not care specifically about the map ρ, we call V itself a representation of G. Since the map ρ induces on V the structure of a G-module, V is also called a G-module. So there is the correspondence between representations of a group G and $\mathbb{C}[G]$-modules. The terms G-module and the representation of G may be used interchangeably. In particular, if $dim(V) = d$ then $GL(V) = GL(d, \mathbb{C})$ such that

$$GL(d, \mathbb{C}) = \{X \in MAT_d(\mathbb{C}) \mid det(X) \neq 0\}.$$

where $MAT_d(\mathbb{C})$ is a complex matrix algebra of dimension d which is the set of all $d \times d$ matrices with complex entries.

Definition 2.15. *A subrepresentation (or submodule) of V is an invariant subspace W of*

V,i.e.,

$$w \in W \Rightarrow gw \in W \textit{ for all } g \in G.$$

$W = V$ and $W = \{0\}$ are called trivial subrepresentations of V. The representation of G can be decomposed into irreducible representations of G. The corresponding result is called Maschke's Theorem:

Theorem 2.2. *(Maschke) Let G be a finite group and let V be a representation of G. Then*

$$V = \rho_1 \oplus \rho_2 \oplus \cdots \rho_k$$

where each ρ_i is an irreducible subrepresentation of V.

We restate the Maschke Theorem using G-modules and full matrix algebras.

Theorem 2.3. *Let G be any finite group. Then*

$$\mathbb{C}[G] \cong MAT_{d_1}(\mathbb{C}) \times \cdots \times MAT_{d_k}(\mathbb{C}).$$

If G is a finite abelian of order n, then $\mathbb{C}[G] = \mathbb{C}^n$. We take a look at the symmetric group for illustrating the representation of a finite group. The symmetric group (denoted by $S(n)$) consists of all bijections from $\{1, \cdots, n\}$ to itself, multiplication being defined as composition. The elements of the symmetric group are called permutations. For a permutation $\sigma \in S(n)$, we use the cycle notation. (more details will be in the next section). Let $G = S(3) = \{e,\ (12),\ (13),\ (23),\ (123),\ (132)\ \}$. There are three irreducible representations of $S(3)$, $\rho_1,\ \rho_2,\ \rho_3$ as follows.

$$\rho_1 : \sigma \in S(3) \rightarrow 1, \quad \text{in } V_1 = \mathbb{C},$$

more explicitly, $\rho_1(e) = 1,\ \rho_1((12)) = 1,\ \rho_1((13)) = 1,\ \rho_1((23)) = 1,\ \rho_1((123)) = 1,$

and $\rho_1((132)) = 1$. This representation is called the trivial representation. In general, the representation $\rho(g) = 1$ for all $g \in G$ is called the trivial representation. Since $dim(V_1) = 1$, it clearly is an irreducible representation.

$$\rho_2 : \sigma \in S(3) \to sgn(\pi), \quad \text{in } V_2 = \mathbb{C},$$

more explicitly, $\rho_2(e) = 1,\ \rho_2((12)) = -1,\ \rho_2((13)) = -1,\ \rho_2((23)) = -1,\ \rho_2((123)) = 1$, and $\rho_2((132)) = 1$. This representation is called the sign representation and it clearly is irreducible representation since $dim(V_2) = 1$.

$$\rho_3 : \sigma \in S(3) \to GL_2(\mathbb{C}), \quad \text{in } V_3 = \mathbb{C}^2,$$

such that

$$\begin{aligned}
\rho_3(e) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \rho_3((123)) &= \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \\
\rho_3((132)) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, & \rho_3((1\ 2)) &= \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \\
\rho_3((1\ 3)) &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, & \rho_3((2\ 3)) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

It is easy to check that $\rho_3(gh) = \rho_3(g)\rho_3(h)$ for all $g,\ h \in S(3)$ and that it is 2-dimensional irreducible representation of $S(3)$. By Theorem (2.3), $\mathbb{C}[S(3)] \cong \mathbb{C} \times \mathbb{C} \times MAT_2(\mathbb{C})$ and it can be expressed as a subalgebra of $MAT_4(\mathbb{C})$.

Proposition 2.4. *Let G be a finite group and let V be a representation of G such that $V = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$ where $\rho_i,\ i \leq i \leq k$, is an irreducible representation. Any* *representation K of G can be expressed as follows:*

$$K = d_1\rho_1 \oplus d_2\rho_2 \oplus \cdots \oplus d_k\rho_k,$$

where $d_1, \cdots, d_k$ are the unique non-zero complex numbers.

Proposition 2.5. *Let $(V, \rho_1),\ (W, \rho_2)$ be $\mathbb{C}[G]$-modules. Then a linear mapping $\phi : V \to W$ is a G-module homomorphism if and only if it commutes with the action of G, i.e. $\rho_2(\phi(x)) = \phi(\rho_1(x))$ for all $\pi \in G$.*

Proposition 2.6. *(Schur's Lemma) Let $V,\ W$ be irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism then ϕ is an isomorphism or $\phi = 0$. Moreover if $V = W$, then $\phi = \lambda \cdot I$ where $\lambda \in \mathbb{C}$ and I is identity.*

Definition 2.16. *Let ρ be the representation of a finite group G on V. The character of a representation ρ (denoted by χ_ρ; when we do not care about ρ we will also write χ_V) is a complex-valued function*

$$\chi_V : G \to \mathbb{C}$$

such that $\chi_V(g) = Tr(\rho(g))$ where $Tr(\rho(g))$ is the trace of the matrix $\rho(g)$.

The character of an irreducible representation is called an irreducible character. Characters play a major role in the theory of group representations since a representation is defined in the unique way by its characters.

Proposition 2.7. *Let V and W be representations of G.*

1. *Every character of the direct sum $\chi_{V\oplus W}$ can be represented in the unique way as $\chi_V + \chi_W$ where χ_V and χ_W are characters of V and W, respectively. ($\chi_{V\oplus W} = \chi_V + \chi_W$).*

2. *Every character of the tensor product $\chi_{V\otimes W}$ can be represented in the unique way as $\chi_V \cdot \chi_W$ where χ_V and χ_W are characters of V and W, respectively. ($\chi_{V\otimes W} = \chi_V \cdot \chi_W$).*

3. *Every character of the dual V^* can be represented in the unique way as $\overline{\chi_V}$ where χ_V is character of V. ($\chi_{V^*} = \overline{\chi_V}$).*

Let us consider again $G = S(3)$. There are three irreducible representations $\rho_1,\ \rho_2,\ \rho_3$ of $S(3)$. By the definition of characters, we can make following table:

	(e)	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
χ_{ρ_1}	1	1	1	1	1	1
χ_{ρ_2}	1	-1	-1	-1	1	1
χ_{ρ_3}	2	0	0	0	-1	-1

Definition 2.17. *A class function on G is a function*

$$f : G \to \mathbb{C}$$

such that $f(hgh^{-1}) = f(g)$ for all $h,\ g \in G$.

Since the trace is invariant under conjugacy, we obtain $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$ for all $g,\ h \in G$. So χ_ρ is a class function.

Definition 2.18. *Let $\alpha, \beta : G \to \mathbb{C}$ be class functions. A Hermitian inner product of α and β is defined as*

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g\in G} \alpha(g)\overline{\beta(g)}.$$

A character table is a table which represents values of characters on conjugacy classes. The character table for $S(3)$ is

	(e)	(1 2)	(1 2 3)
χ_{ρ_1}	1	1	1
χ_{ρ_2}	1	-1	1
χ_{ρ_3}	2	0	-1

We make the following observations from the character table for $S(3)$ and we provide analogous statements for $S(n)$ without proofs. Let k be the number of irreducible representations for $S(n)$.

1. The number of irreducible representations of $S(3)$ is equal to the the number of conjugacy classes of $S(3)$. This is general true for $S(n)$.

2. $\chi_{\rho_1}(1),\ \chi_{\rho_2}(1),\ \chi_{\rho_3}(1)$ are the dimensions representations of $\rho_1,\ \rho_2,\ \rho_3$ respectively. In general, $\chi_\rho(1) = dim(\rho)$.

3. $\chi_{\rho_1}(1)^2 + \chi_{\rho_2}(1)^2 + \chi_{\rho_3}(1)^2 = 3!$. In general, if $d_1,\ d_2,\ \cdots,\ d_k$ are dimensions of the representations $\rho_1,\ \rho_2,\ \cdots, \rho_k$, then $\sum\limits_{i=1}^{k} d_i^2 = n!$.

4. $\sum\limits_{\sigma\in S(3)} \chi_i(\sigma)\overline{\chi_j(\pi)} = \delta_{ij}$. In general, $\langle \chi_i, \chi_j \rangle = \sum\limits_{\sigma\in S(n)} \chi_i(\sigma)\overline{\chi_j(\pi)} = \delta_{ij}$. It means that the character table holds row orthogonality and therefore is always a unitary matrix.

5. $\sum\limits_{m=1}^{3} \chi_m(\sigma_i)\chi_m(\sigma_j) = \delta_{ij}$ for $\sigma_i,\ \sigma_j \in S(3)$. In general $\sum\limits_{m=1}^{k} \chi_m(\sigma_i)\chi_m(\sigma_j) = \delta_{ij}$ for $\sigma_i,\ \sigma_j \in S(n)$. It means that the character table holds column orthogonality.

6. Notice that $\langle \chi_1, \chi_1 \rangle = \langle \chi_2, \chi_2 \rangle = \langle \chi_3, \chi_3 \rangle = 1$, and the representations $\rho_1,\ \rho_2,\ \rho_3$ are irreducible representations. In general, $\langle \chi_i, \chi_i \rangle = 1$ if and only if ρ_i is irreducible representation.

7. $\frac{1}{3!}\sum_{i=1}^{3}\chi_i(e) = 1$, $\frac{1}{3!}\sum_{i=1}^{3}\chi_i((1\ 2)) = 3$, and $\frac{1}{3!}\sum_{i=1}^{3}\chi_i((1\ 2\ 3)) = 2$. In general, for a fixed $\sigma \in S(n)$, $\frac{1}{n!}\sum_{i=1}^{k}\chi_i(\sigma) = |c_\sigma|$, where $|c_\sigma|$ is the number of elements in the corresponding conjugacy class.

8. $\chi_1(e) + \chi_2(e) + 2\chi_2(e) = 3!$ and $\chi_1(\sigma) + \chi_2(\sigma) + 2\chi_2(\sigma) = 0$, for $\sigma \neq e \in S(3)$. In general, $\sum_{i=1}^{k} d_i\chi_i(e) = n!$ and for $\pi \neq e \in S(n)$,$\sum_{i=1}^{k} d_i\chi_i(\pi) = 0$ where d_i is the dimension of irreducible representations of $S(n)$.

Proposition 2.8. *Let $\sigma \in S(n)$. Then $|\chi_\rho(\sigma)| = k$ if and only if $\rho(\sigma) = c \cdot I_k$ where $c = \frac{\chi_\rho(\sigma)}{k}$ and I_k is $k \times k$ identity matrix.*

We state Theorem for characters of a finite group.

Theorem 2.4. *If $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$ where $\rho_i,\ 1 \leq i \leq n$ are irreducible representations, then every character χ_{ρ_i}, can be expressed as follows:*

$$\chi = \sum_{i=1}^{n} k_i \chi_{\rho_i},$$

where k_i are nonnegative integers.

2.5 Irreducible representations of the finite symmetric group $S(n)$

In this section, we consider the finite symmetric group $S(n)$. We cite without proofs from [7], [8], [15], [32] and [34]. For a permutation $\sigma \in S(n)$, we use the cycle notation. A k-cycle is a cycle of length k. The cycle type of σ has a form $(1^{k_1}, 2^{k_2}, \cdots, n^{k_n})$ where k_i is the number of cycles of length k. For example, if $\sigma = (1\ 2\ 3)(4\ 5) \in S(5)$ then σ consists of a 3-cycle and a 2-cycle. σ has the cycle type form $(1^0, 2^1, 3^1, 4^0, 5^0)$. The cycle type can be expressed as a partition.

Definition 2.19. *A partition λ of n (denoted by $\lambda \vdash n$) is a sequence*

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m),\ 1 \leq i \leq m$$

where λ_i are weakly decreasing and $\sum_{i=1}^{m} = n$.

$\lambda = (3, 2)$ is the partition for $\sigma = (1\ 2\ 3)(4\ 5)$. For $n = 5$, we obtain following partitions: $(5),\ (4, 1),\ (3, 2),\ (3, 1, 1),\ (2, 2, 1),\ (2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$.

Definition 2.20. *Let G be group. Two elements g and h are conjugate if $g = khk^{-1}$ for some $k \in G$. The conjugacy class of g is the set of elements conjugate to an element g.*

If $\sigma_1 \in S(n)$ is any permutation and it can be expressed as $\sigma_1 = (i_1, \cdots, i_k) \cdots (i_m, \cdots, i_n)$, then for any $\sigma_2 \in S(n)$

$$\sigma_2 \sigma_1 \sigma_2^{-1} = (\sigma_2(i_1), \cdots, \sigma_2(i_k)) \cdots (\sigma_2(i_m), \cdots, \sigma_2(i_n)).$$

It follows that the cycle type is preserved under conjugacy and we obtain following Theorem:

Theorem 2.5. *Two permutations $\sigma_1,\ \sigma_2 \in S(n)$ are conjugate if and only if they have the same cycle type. The conjugacy classes can be labeled by integers $m_1, \cdots, m_n$ with $m_j \geq 0$, $\sum_{j=1}^{n} jm_j = n$ where $(m_1, \cdots, m_n)$ is from the cycle type $(1^{m_1}, \cdots, n^{m_n})$ of two permutations.*

Clearly, there is one-to-one correspondence between the conjugacy classes of $S(n)$ and the partitions of n. We show it below with a partition and a representative element of each conjugacy class. For example,

$$
\begin{aligned}
\lambda_1 = (5) &\quad \leftrightarrow \quad (1\ 2\ 3\ 4\ 5) \text{ (a 5-cycle)} \\
\lambda_2 = (4,1) &\quad \leftrightarrow \quad (1\ 2\ 3\ 4)(5) \text{ (a 4-cycle and a 1-cycle)} \\
\lambda_3 = (3,2) &\quad \leftrightarrow \quad (1\ 2\ 3)(4\ 5) \text{ (a 3-cycle and a 2-cycle)} \\
\lambda_4 = (3,1,1) &\quad \leftrightarrow \quad (1\ 2\ 3)(4)(5) \text{ (a 3-cycle and two 1-cycles)} \\
\lambda_5 = (2,2,1) &\quad \leftrightarrow \quad (1\ 2)(3\ 4)(5) \text{ (two 2-cycle and a 1-cycle)} \\
\lambda_6 = (2,1,1,1) &\quad \leftrightarrow \quad (1\ 2)(3)(4)(5) \text{ (a 2-cycle and three 1-cycles)} \\
\lambda_7 = (1,1,1,1,1) &\quad \leftrightarrow \quad (1)(2)(3)(4)(5) \text{ (five 1-cycles)}
\end{aligned}
$$

There are 7 conjugacy classes in $S(5)$ and we can compute the number of elements of a conjugacy class in the following Theorem:

Theorem 2.6. *If $\sigma \in S(n)$ has a form $(1^{m_1}, 2^{m_2}, \cdots, n^{m_n})$, then the number of elements in each conjugacy class is*

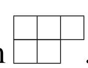
$$\frac{n!}{1^{m_1}m_1!2^{m_2}m_2!\cdots n^{m_n}m_n!}.$$

For $S(5)$, we compute them as follows:

$$\begin{array}{rcl}
(1\ 2\ 3\ 4\ 5) & \leftrightarrow & (1^0, 2^0, 3^0, 4^0, 5^1) \rightarrow \dfrac{5!}{1^0 0! 2^0 0! 3^0 0! 4^0 0! 5^1 1!} = 24 \\
(1\ 2\ 3\ 4)(5) & \leftrightarrow & (1^1, 2^0, 3^0, 4^1, 5^0) \rightarrow \dfrac{5!}{1^1 1! 2^0 0! 3^0 0! 4^1 1! 5^0 0!} = 30 \\
(1\ 2\ 3)(4\ 5) & \leftrightarrow & (1^0, 2^1, 3^1, 4^0, 5^0) \rightarrow \dfrac{5!}{1^0 0! 2^1 1! 3^1 1! 4^0 0! 5^0 0!} = 20 \\
(1\ 2\ 3)(4)(5) & \leftrightarrow & (1^2, 2^0, 3^1, 4^0, 5^0) \rightarrow \dfrac{5!}{1^2 2! 2^0 0! 3^1 1! 4^0 0! 5^0 0!} = 20 \\
(1\ 2)(3\ 4)(5) & \leftrightarrow & (1^1, 2^2, 3^0, 4^0, 5^0) \rightarrow \dfrac{5!}{1^1 1! 2^2 2! 3^0 0! 4^0 0! 5^0 0!} = 15 \\
(1\ 2)(3)(4)(5) & \leftrightarrow & (1^3, 2^1, 3^0, 4^0, 5^0) \rightarrow \dfrac{5!}{1^3 3! 2^1 1! 3^0 0! 4^0 0! 5^0 0!} = 10 \\
(1)(2)(3)(4)(5) & \leftrightarrow & (1^5, 2^0, 3^0, 4^0, 5^0) \rightarrow \dfrac{5!}{1^5 5! 2^0 0! 3^0 0! 4^0 0! 5^0 0!} = 1
\end{array}$$

We see that the conjugacy classes of $S(n)$ can be matched with the partition of n from the previous section. In this section, we discuss Young diagrams and Young tableaux which is another way to show correspondence for the conjugacy classes of $S(n)$

Definition 2.21. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. A Young diagram of shape λ consists of k left-justified boxes with row i containing λ_i boxes for $1 \leq i \leq k$.*

Note that the symbol λ stands for the partition and its shape. For example, the partition $\lambda = (3, 2)$ has a Young diagram . So Young diagrams for $n = 5$ are in Figure 2.1. Clearly, there is one-to-one correspondence between Young diagrams for $n = 5$ and the partitions for $n = 5$. So the number of conjugacy classes of $S(n)$ is the the number of partitions of n and the number of Young diagrams on n.

Definition 2.22. *Suppose $\lambda \vdash n$. A Young tableau of shape λ (or λ-tableau)(denoted by T^{λ}) is a Young diagram filling the numbers $(1, 2, \cdots, n)$ to the n boxes bijectively without repetition.*

If $\lambda = (2, 2, 1)$ then there are 120 possible Young tableaux for $T^{(2,2,1)}$.

$$\lambda_1 = (5) \quad \Longleftrightarrow$$
$$\lambda_2 = (4, 1) \quad \Longleftrightarrow$$
$$\lambda_3 = (3, 2) \quad \Longleftrightarrow$$
$$\lambda_4 = (3, 1, 1) \quad \Longleftrightarrow$$
$$\lambda_5 = (2, 2, 1) \quad \Longleftrightarrow$$
$$\lambda_6 = (2, 1, 1, 1) \quad \Longleftrightarrow$$
$$\lambda_7 = (1, 1, 1, 1, 1) \quad \Longleftrightarrow$$

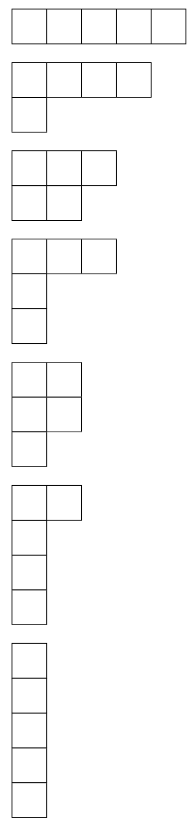


Figure 2.1: Young diagrams for $n = 5$

Definition 2.23. *Suppose $\lambda \vdash n$. T^λ is called standard if the numbers increase in each row and each column. T^λ is called semistandard if the numbers weakly increase in each row and strictly in each column. Let $f(T^\lambda)$ be the number of standard Young tableaux for λ.*

The number of standard Young tableaux for λ denotes by $f(T^\lambda)$. For $\lambda = (2,2,1)$, then $f(T^{(2,2,1)}) = 5$ since

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

The number of standard tableaux is important since it is the dimension of the irreducible representation associated to $S(n)$. We can find it by Hook Length formula:

Theorem 2.7. *If $\lambda \vdash n$, then*

$$f(T^\lambda) = \frac{n!}{\prod_b h_b}$$

where the product is over all boxes b of λ and h_b is the hook-length [2] *of b.*

By the Hook length formula, we can make Young graph in Figure 2.2.

We state following Theorem:

Theorem 2.8.

$$\sum_{\lambda \vdash n} f(T^\lambda)^2 = n!$$

Clearly, $1^2 + 4^2 + 5^2 + 6^2 + 4^2 + 1^1 = 5!$.

Since the number of conjugacy classes of $S(n)$ equals to the number of Young diagrams on n, we can construct all the irreducible representations of $S(n)$ over $\mathbb{C}$ using the Specht module.

[2]The number of boxes directly to the right or below b (including b itself).

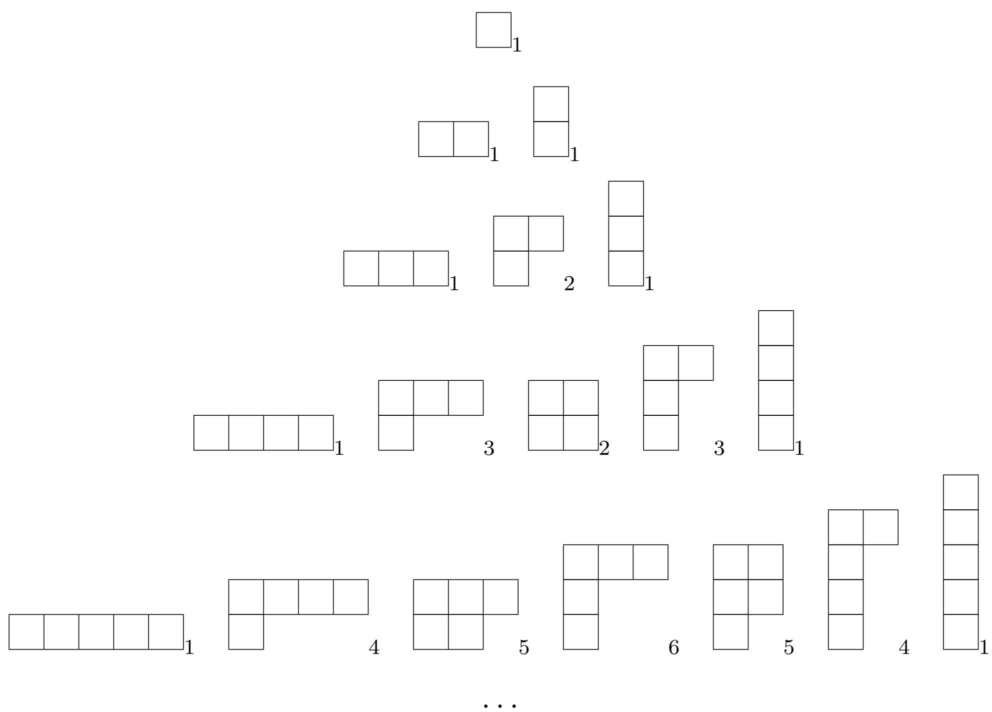


Figure 2.2: Young graph by Hook length formula. Notice that the subscripts tell us the dimension of diagrams

Definition 2.24. *Two λ-tableaux T^{λ_1} and T^{λ_2} are row equivalent if they contain the same numbers for each corresponding rows. A tabloid of shape of λ (or λ-tabloid) (denoted by $\{T^{\lambda}\}$ is set of all equivalent of λ-tableaux.*

If $T^{(2,1)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, then

$$\{T^{(2,1)}\} = \{ \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \} = \overline{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}$$

Tabloids are denoted by bar notation. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ then the number of λ-tabloids is $\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_k!}$ since the number of tableaux of a row equivalence class is $\lambda_1!\lambda_2!\cdots\lambda_k!$.

Definition 2.25. *Suppose $\lambda \vdash n$. The permutation module of λ (denoted by M^{λ}) is*

$$M^{\lambda} = \mathbb{C}\{T_1^{\lambda}, \cdots, T_k^{\lambda}\}$$

where $T_1^{\lambda}, \cdots, T_k^{\lambda}$ is complete list of λ-tabloids.

There are 10 λ-tabloids for $n = 3$:

$$\{ \quad \overline{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}, \quad \overline{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}, \quad \overline{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}}, \quad \overline{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}}, \quad \overline{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} \quad \}.$$

If λ-tableau is a standard, then the corresponding tabloid is also standard.

The symmetric group $\pi \in S(n)$ acts on tabloids by

$$\pi\{T^{\lambda}\} = \{\pi T^{\lambda}\}.$$

This action extends to an $S(n)$-module. We define two Young subgroups of the symmetric groups $S(n)$:

$$\begin{aligned} R_{T^\lambda} &= \{g \in S(n) : \ g \text{ preserves each row } \} \\ C_{T^\lambda} &= \{g \in S(n) : \ g \text{ preserves each column } \}. \end{aligned}$$

Then we define a_{T^λ} and b_{T^λ} in $\mathbb{C}[S(n)]$ as follows:

$$a_{T^\lambda} = \sum_{p \in R_{T^\lambda}} p, \qquad b_{T^\lambda} = \sum_{q \in C_{T^\lambda}} sgn(q)q.$$

Definition 2.26. *For T^λ, the associated polytabloid (denoted by v_{T^λ}) is defined by* $v_{T^\lambda} = b_{T^\lambda}\{T^\lambda\} = \sum_{q \in C_{T^\lambda}} sgn(q)\{qT^\lambda\}$.

The product $c_{T^\lambda} = b_{T^\lambda} a_{T^\lambda}$ is called Young symmetrizers. Now we define the Specht module.

Definition 2.27. *Suppose $\lambda \vdash n$. The Specht module (denoted by S^λ) is the submodule of M^λ spanned by polytabolids v_{T^λ} where T^λ varies over all numberings of λ.*

Note that S^λ is preserved by $S(n)$ since $\pi v_{T^\lambda} = v_{\pi T^\lambda}$ for all T^λ and all $\pi \in S(n)$. We obtain the following proposition:

Proposition 2.9. *The Specht module S^λ are cyclic modules generated by polytabolids v_{T^λ}.*

We take look at special cases ($\lambda = (n), \ \lambda = (1, \cdots, 1) = (1^n), \ \lambda = (n-1, 1)$) for The Specht module S^λ. If $\lambda = (n)$ then there is only one polytabloid $\overline{\overline{\boxed{1 \cdots n}}}$ so S^λ is the trivial representation of $S(n)$ and $dim(S^{(n)}) = 1$. If $\lambda = (1^n)$ then each equivalence class $\{T^\lambda\}$ consists of a single tableau and the action of $S(n)$ is preserved. So S^λ is the sign representation of $S(n)$ and $dim(S^{(1^n)}) = 1$. For $\lambda = (n-1, 1)$ λ-tabloid is uniquely determined by the element of second row. $dim(S^{(n-1,1)}) = n-1$. When $n = 3$, these three cases give all the irreducible representations of $S(3)$. We state the following Theorem for general case of n.

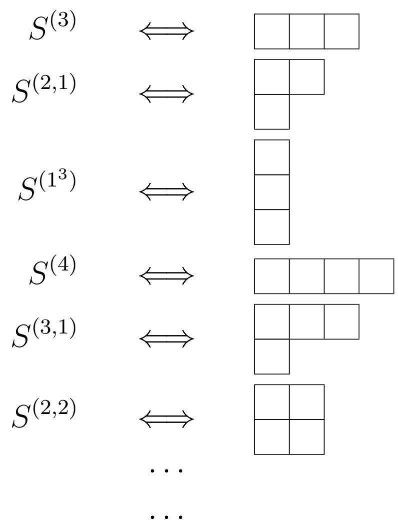


Figure 2.3: The correspondence between Specht modules and Young diagrams

Theorem 2.9. *Suppose $\lambda \vdash n$. The Specht module, S^λ, is an irreducible representation of $S(n)$ in $\mathbb{C}$. Every irreducible representation of $S(n)$ is isomorphic to exactly one S^λ.*

By Theorem (2.9), we can make correspondence in Figure 2.3.

The dimension of an irreducible representation of $S(n)$ is the number of standard λ-tableau on n. We summarize it following Theorem:

Theorem 2.10. *Let $\lambda \vdash n$.*

1. $dim(S^\lambda) = f(T^\lambda)$,

2. $\sum\limits_{\lambda \vdash n} f(T^\lambda)^2 = n!$,

3. $\{v_{T^\lambda} : T^\lambda$ *is a standard λ-tableau* $\}$ *is a basis for* S^λ.

There is the relationship between M^λ and S^λ.

Definition 2.28. *Suppose $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, \cdots, \mu_m)$ are partition of n. Then*

λ dominates μ (denoted by $\lambda \trianglerighteq \mu$) if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for all $i \geq 1$.

Note that if $\lambda,\ \mu \vdash n$ with $\lambda \trianglerighteq \mu$, then $\lambda \geq \mu$.

Proposition 2.10. *The permutation module M^λ can be decomposed into the direct sum of the Specht modules, S^λ;*

$$M^\lambda = \bigoplus_{\lambda \geq \mu} K_{\lambda,\mu} S^\lambda$$

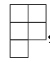
with $K_{\mu,\mu} = 1$.

The coefficient $K_{\lambda,\mu}$ is the number of semistandard λ-tableaux with content μ. We call this formula Young's Rule. If $\mu = (1^n)$ then $K_{\lambda,\mu}$ is the number of standard λ-tableaux. For example,

$$M^{(1,1,1,1,1)} = S^{(1,1,1,1,1)} \oplus S^{(1,1,1,1,1)} \oplus 4S^{(2,1,1,1)} \oplus 5S^{(2,2,1)} \oplus 6S^{(3,1,1)} \oplus 4S^{(4,1)} \oplus S^{(5)}$$

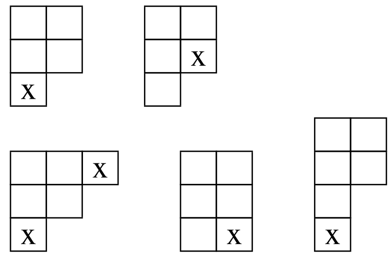
The branching rule describes irreducible representations S^λ of $S(n)$ under restricting or inducing to $S(n-1)$ or $S(n+1)$. The restricting or inducing are simply either removing or adding a box to the Young diagram for λ.

Definition 2.29. *Supposed $\lambda \vdash n$ is a Young diagram. An inner box of λ (denoted by λ^-) is a box $(i, j) \in \lambda$ which we can remove from the Young diagram of a partition. An outer box of λ (denote by λ^+) is a box $(i, j) \notin \lambda$ which we can add to the Young diagram of a partition.*

For example, if $\lambda = (2,2,1) \leftrightarrow$ , then

λ^- :

λ^+ :



Note that x indicates either removing or adding a box. From the example, we restrict an irreducible representation $S^{(2,2,1)}$ of $S(5)$ to $S(4)$ (denoted by $S^\lambda \downarrow^{S(n)}_{S(n-1)}$ for general) and

$$S^{(2,2,1)} \downarrow^{S(5)}_{S(4)} \cong \ S^{(2,2)} \oplus S^{(2,1,1)}.$$

We induce an irreducible representation $S^{(2,2,1)}$ of $S(5)$ to $S(6)$ (denoted by $S^\lambda \uparrow^{S(n+1)}_{S(n)}$ for general) and

$$S^{(2,2,1)} \uparrow^{S(6)}_{S(5)} \cong \ S^{(3,2,1)} \oplus S^{(2,2,2)} \oplus S^{(2,2,1,1)}.$$

Now we take Branching Rule of $S(n)$:

Theorem 2.11. *Suppose $\lambda \vdash n$. Then*

1. $S^\lambda \downarrow^{S(n)}_{S(n-1)} \cong \bigoplus\limits_{\lambda^-} S^{\lambda^-}$,
2. $S^\lambda \uparrow^{S(n+1)}_{S(n)} \cong \bigoplus\limits_{\lambda^+} S^{\lambda^+}$,
3. $f(T^\lambda) = \sum\limits_{\lambda^-} f(T^{\lambda^-})$.

Theorem (2.11) (c) tells us about the relationship of dimensions between S^λ of $S(n)$ and S^{λ^-} of $S(n-1)$. For example $f(T^{(2,2,1)}) = 5$ since $f(T^{(2,2)}) = 2$ and $f(T^{(2,1,1)}) = 3$.

The Littlewood-Richardson Rule play a crucial role in the decomposition of certain induced representations in the representation theory of the symmetric group. The Littlewood-

Richardson Rule is

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

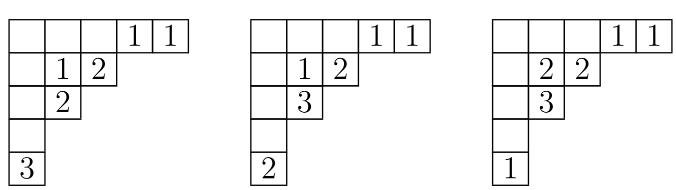
It describes the Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$ when decomposing a product of two Schur (see Appendix G) functions as a linear combination of other Schur functions. It is a natural numbers and it depends on three partitions λ, μ, and ν. Unless $\nu \subseteq \lambda$ and $\subseteq \mu$, we have $c^\nu_{\lambda,\mu} = 0$. Here $\nu \subseteq \lambda, \mu$ means that for any i, $\nu_i \geq \lambda_i$ $\nu_i \geq \mu_i$ holds. Moreover, if $|\lambda| + |\mu| \neq |\nu|$, then $c^\nu_{\lambda,\mu} = 0$. If all these conditions are satisfied, $c^\nu_{\lambda,\mu}$ can be calculated with a skew Young tableaux of shape ν/λ and of weight μ. The skew Young diagram ν/λ denotes the diagram obtained by removing the square of λ sitting in ν such that the left upper squares of λ and ν fit in. For $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$, we write the numbers μ_1 $1's$, μ_2 $2's \cdots, \mu_k$ $k's$ in the skew Young diagram ν/λ satisfying the following conditions:

(a) The nonempty entire first line must be filled with entries 1

(b) For each row, the numbers are weakly decreasing from left to right

(c) For each column, the numbers are strictly increasing from top to bottom

Then $c^\nu_{\lambda,\mu}$ is the number of ways to write down the numbers μ_1 $1's$, μ_2 $2's, \cdots, \mu_k$ $k's$ in the skew Young diagram ν/λ. For example, we will calculate $c^\nu_{\lambda,\mu}$ for $\nu = (5,3,2,1,1)$, $\lambda = (3,1,1,1)$, $\mu(3,2,1)$:



Thus, $c^\nu_{\lambda,\mu} = 3$.

3. WREATH PRODUCT OF THE FINITE SYMMETRIC GROUP $S(\infty)$ WITH Z_2

3.1 Wreath product of the infinite symmetric group $S(\infty)$ with Z_2

The definition of wreath product can be found in many sources. We follow below the version given in [42] which is one of most detailed and clearly written.

Let P, Q be sets and A, B be subgroups of permutation group $S(P)$, $S(Q)$ respectively. The set of all functions from P to B, B^P, is also a group with natural multiplication $\phi\psi(i) = \phi(i)\psi(i)$ where ϕ, ψ in B^p, i in P. Each pair (a, ϕ), $a \in A$, $\phi \in B^P$, defines a permutation on the set $P \times Q = \{(i, j) : i \in P, j \in Q\}$ as

$$(a, \phi)(i, j) = (ai, \phi(i)j).$$

From here it is immediate to see that

$$(a_1, \phi_1)(a_2, \phi_2) = (a_1a_2, (\phi_1(a_2)\phi_2)).$$

The identity element is

$$E = (e, \epsilon)$$

where e is the identity in A and ϵ is the function that maps each element of P to the identity of B.

The inverse of (a, ϕ) is

$$(a, \phi)^{-1} = (a^{-1}, (\phi a^{-1})^{-1}).$$

The obtained group is called **the wreath product of A with B and will be denoted by** $A[B]$. The sets $\{(a, \epsilon) \mid a \in A\}$ and $\{(e, \phi) \mid \phi \in B^P\}$ are subgroups of $A[B]$ isomorphic to A and B^P, respectively. Moreover the subgroup $\{(e, \phi)\}$ is a normal subgroup of $A[B]$.

Since Z_2 is isomorphic to $S(2)$, we will use $S(2)$ instead of Z_2. We will describe the wreath product of $S(\infty)$ with Z_2 (denoted by $S(\infty)[Z_2]$).

Let

$$\begin{aligned} A &= S(\infty) \qquad P = \{1, 2, 3, \cdots\}, \\ B &= S(2) \qquad\quad Q = \{1, 2\}. \end{aligned}$$

Let $B^P = \{\phi : P \to S(2)\}$, For any $i \in \mathbb{N}$, $\quad \phi(i) = \begin{cases} (1)(2), \\ (1\ 2). \end{cases}$

The composition of two permutations in $S(\infty)[S(2)]$ is as following $(a_1, \phi_1)(a_2, \phi_2)(i, j) = (a_1, \phi_1)(a_2 i, \phi_2(i)j) = (a_1 a_2 i, \phi_1(a_2 i)[\phi_2(i)j]), \quad i \in P, \ j \in \{1, 2\}$. The identity element and the inverse are defined by

$$E = (e, \epsilon), \quad e = e_A, \epsilon(i) = e_B,$$

$$(a, \phi)^{-1} = (a^{-1}, (\phi(a^{-1}))^{-1}).$$

Notice also that the wreath product of $S(\infty)$ with Z_2 is the semidirect product of $S(\infty)$ and $K = \prod_{i=1}^{\infty}(Z_2)_i$, where K is the full direct product of infinitely many copies of Z_2. This product $K = \prod_{i=1}^{\infty}(Z_2)_i$ is endowed with Tychonoff topology, i.e. every neighborhood of identity in it is the form $\prod_{i=1}^{\infty} O_i$ where O_i is a neighborhood of the identity in Z_2 and $O_i = Z_2$ with exception of a finite set of indices.

To better understand the construction of this wreath product of $S(\infty)$ with Z_2 we will first consider the wreath product of $S(n)[Z_2], \ n \in \mathbb{N}$.

Example 1

The wreath product of $S(1)$ with Z_2 has 2 elements:

$$S(1)[Z_2] = \{(a, \phi) \mid a \in S(1),\ \phi : S(1) \to S(2)\}.$$

We will consider its matrix representation. Let

$$A = S(1) \qquad B = S(2)$$

$$P = \{1\} \qquad Q = \{1,\ 2\}$$

$$B^P : P \to S(2)$$

Let $e = (1)(2),\ s = (1\ 2)$. Then B^P has two elements:

$$f_1(1) = e, \quad f_2(1) = s$$

respectively. The action of the wreath product can be described as in the case of f_1:

$$\begin{aligned} (e, f_1)(1,1) &= (e1, f_1(1)1) = (1,1) \\ (e, f_1)(1,2) &= (e1, f_1(1)2) = (1,2) \end{aligned}$$

The corresponding matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

And in the case of f_2 we have

$$\begin{aligned} (e, f_2)(1,1) &= (e1, f_2(1)1) = (1,2) \\ (e, f_2)(1,2) &= (e1, f_2(1)2) = (1,1) \end{aligned}$$

with the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The subgroup $S(1)[Z_2]$ is isomorphic to a subgroup of $GL(2, \mathbb{R})$ which elements are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 2

The wreath product of $S(2)$ with Z_2 has 8 elements:

$$S(2)[Z_2] = \{(a, \phi) \mid a \in S(2),\ \phi : S(2) \to S(2)\}$$

We will consider its matrix representation. Let

$$A = S(2) \qquad B = S(2)$$

$$P = \{1,\ 2\} \qquad Q = \{1,\ 2\}$$

$$B^P : P \to S(2)$$

Let $e = (1)(2)$ $s = (1\ 2)$ Then consider B^P has four elements:

$$f_1(1) = e, \quad f_1(2) = e$$

$$f_2(1) = e, \quad f_2(2) = s$$

$$f_3(1) = s, \quad f_3(2) = e$$

$$f_4(1) = s, \quad f_4(2) = s$$

The action of the wreath product can be described as in the case of (e, f_1):

$$\begin{aligned}
(e, f_1)(1,1) &= (e1, f_1(1)1) = (1,1)\\
(e, f_1)(1,2) &= (e1, f_1(1)2) = (1,2)\\
(e, f_1)(2,1) &= (e2, f_1(2)1) = (2,1)\\
(e, f_1)(2,2) &= (e2, f_1(2)2) = (2,2)
\end{aligned}$$

The corresponding matrix is

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right].$$

The action of the wreath product can be described as in the case of (e, f_2):

$$\begin{aligned}
(e, f_2)(1,1) &= (e1, f_2(1)1) = (1,1)\\
(e, f_2)(1,2) &= (e1, f_2(1)2) = (1,2)\\
(e, f_2)(2,1) &= (e2, f_2(2)1) = (2,2)\\
(e, f_2)(2,2) &= (e2, f_2(2)2) = (2,1)
\end{aligned}$$

The corresponding matrix is

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The action of the wreath product can be described as in the case of (e, f_3):

$$\begin{aligned} (e, f_3)(1,1) &= (e1, f_3(1)1) = (1,2) \\ (e, f_3)(1,2) &= (e1, f_3(1)2) = (1,1) \\ (e, f_3)(2,1) &= (e2, f_3(2)1) = (2,1) \\ (e, f_3)(2,2) &= (e2, f_3(2)2) = (2,2) \end{aligned}$$

The corresponding matrix is

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The action of the wreath product can be described as in the case of (e, f_4):

$$\begin{aligned} (e, f_4)(1,1) &= (e1, f_4(1)1) = (1,2) \\ (e, f_4)(1,2) &= (e1, f_4(1)2) = (1,1) \\ (e, f_4)(2,1) &= (e2, f_4(2)1) = (2,2) \\ (e, f_4)(2,2) &= (e2, f_4(2)2) = (2,1) \end{aligned}$$

The corresponding matrix is

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The simple calculation below in the case of (s, f_1)

$$\begin{aligned} (s, f_1)(1,1) &= (s1, f_1(1)1) = (2,1) \\ (s, f_1)(1,2) &= (s1, f_1(1)2) = (2,2) \\ (s, f_1)(2,1) &= (s2, f_1(2)1) = (1,1) \\ (s, f_1)(2,2) &= (s2, f_1(2)2) = (1,2) \end{aligned}$$

provides the following matrix:

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Similarly, in the case of (s, f_2) we have

$$\begin{aligned} (s, f_2)(1,1) &= (s1, f_2(1)1) = (2,1) \\ (s, f_2)(1,2) &= (s1, f_2(1)2) = (2,2) \\ (s, f_2)(2,1) &= (s2, f_2(2)1) = (1,2) \\ (s, f_2)(2,2) &= (s2, f_2(2)2) = (1,1) \end{aligned}$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

Similarly, in the case of (s, f_3) we have

$$\begin{aligned}
(s, f_3)(1,1) &= (s1, f_3(1)1) = (2,2) \\
(s, f_3)(1,2) &= (s1, f_3(1)2) = (2,1) \\
(s, f_3)(2,1) &= (s2, f_3(2)1) = (1,1) \\
(s, f_3)(2,2) &= (s2, f_3(2)2) = (1,2)
\end{aligned}$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Similarly, in the case of (s, f_4) we have

$$\begin{aligned}
(s, f_4)(1,1) &= (s1, f_4(1)1) = (2,2) \\
(s, f_4)(1,2) &= (s1, f_4(1)2) = (2,1) \\
(s, f_4)(2,1) &= (s2, f_4(2)1) = (1,2) \\
(s, f_4)(2,2) &= (s2, f_4(2)2) = (1,1)
\end{aligned}$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

The subgroup $S(2)[Z_2]$ is isomorphic to a subgroup of $GL(4, \mathbb{R})$ which elements are:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

Example 3

The wreath product of $S(3)$ with Z_2 has 48 elements:

$$S(3)[Z_2] = \{(a, \phi) \mid a \in S(3),\ \phi : S(3) \to S(2)\}$$

We will consider its matrix representation. Let

$$A = S(3) = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \qquad B = S(2)$$

$$P = \{1,\ 2,\ 3\} \qquad Q = \{1,\ 2\}$$

$$S(3)[S(2)] = \{(a, \phi) \mid a \in A,\ \phi \in B^P\}$$

$$B^P : P \to S(2)$$

Let $e = (1)(2)$ $s = (1\ 2)$ Then B^P has eight elements:

$$f_1(1) = e, \quad f_1(2) = e, \quad f_1(3) = e$$

$$f_2(1) = e, \quad f_2(2) = e, \quad f_2(3) = s$$

$$f_3(1) = e, \quad f_3(2) = s, \quad f_3(3) = e$$

$$f_4(1) = e, \quad f_4(2) = s, \quad f_4(3) = s$$

$$f_5(1) = s, \quad f_5(2) = e, \quad f_5(3) = e$$

$$f_6(1) = s, \quad f_6(2) = e, \quad f_6(3) = s$$

$$f_7(1) = s, \quad f_7(2) = s, \quad f_7(3) = e$$

$$f_8(1) = s, \quad f_8(2) = s, \quad f_8(3) = s$$

The calculation can be found in Appendix C.

3.2 The relationship between $S(\infty)$ and $S(\infty)[Z_2]$

$S(n)$ is isomorphic to all the square matrices with entries 0 and 1 of order n such that in any row or column the number of $1'$s is exactly one. We can construct the following matrices of $S(n),\ n = 1, 2, 3, \cdots$ below:

$$
\begin{array}{lcl}
S(1) & \Longleftrightarrow & 1 \\
S(2) & \Longleftrightarrow & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
S(3) & \Longleftrightarrow & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\
 & & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\
S(4) & \Longleftrightarrow & 24 \text{ of } 4 \text{ by } 4 \text{ matrices} \\
\cdots & \Longleftrightarrow & \cdots
\end{array}
$$

The example above shows that the group $S(n)[Z_2]$ is isomorphic to the subgroup of $GL(2n, \mathbb{R})$ obtained as follows. We consider a matrix corresponding to an element of $S(n)$ and we substitute each entry of it equal to 1 by either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and any entry equal to 0 by the block $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So we obtain the following matrices of $S(n)[Z_2],\ n = 1, 2, 3, \cdots$ below:

$$S(1)[Z_2] \iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$S(2)[Z_2] \iff \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

$S(3)[Z_2] \iff$ 48 of 6 by 6 matrices can be found in Appendix C.

$\cdots \iff \cdots$

If we have such a matrix and have replacements described above we will get a group of infinite matrices isomorphic to $S(\infty)[Z_2]$.

3.3 Semigroups $\Phi(\infty)$ and $\Phi(\infty)[Z_2]$ generated by $S(\infty)$ and $S(\infty)[Z_2]$

We introduce (following [26]) the semigroup of infinite matrices $\Phi(\infty)$ whose elements are matrices with entries $0,\ 1$ such that in every row and in every column there is at most one element equal to 1. Clearly $S(\infty)$ is subsemigroup of $\Phi(\infty)$. We introduce involution of $\Phi(\infty)$ as $\phi^* = \phi^T$ where ϕ^T means the matrix transpose to ϕ. We can construct following matrices of $\Phi(n),\ n = 1, 2, \cdots$ below:

$$\begin{array}{ccl}
\Phi(1) & \Longleftrightarrow & 1, \quad 0 \\
\Phi(2) & \Longleftrightarrow & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
 & & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \\
\cdots & \Longleftrightarrow & \cdots\cdots
\end{array}$$

We define $\Phi(\infty)[Z_2]$ by considering the matrices from $\Phi(\infty)$ and substituting in such a matrix every 1 by either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and any entry equal to 0 by the block $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We can construct following matrices of $\Phi(n)[Z_2],\ n = 1, 2, 3$:

$$\Phi(1)[Z_2] \iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Phi(2)[Z_2] \iff \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right],$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$\Phi(3)[Z_2] \iff$ 127 of 6 by 6 matrices

3.4 Description of $S(\infty)[Z_2]$ as a group of permutations and its subgroups

The wreath product of infinite symmetric group $S(\infty)$ with Z_2 (denoted by $S(\infty)[Z_2]$) can be described as following way. We represent $\mathbb{N}$ as the union of $\{2i-1, 2i\}$. Consider an arbitrary $\tilde{\sigma}_1 \in S(\infty)$. Next consider the permutation σ_1 defined as

$$\sigma_1(2i-1) = 2\tilde{\sigma}_1(i) - 1, \quad \sigma_1(2i) = 2\tilde{\sigma}_1(i)$$

Notice that the correspondence between $\tilde{\sigma}_1$ and σ_1 is bijective. Now consider a permutation σ_2 on $\mathbb{N}$ such that $\forall i,\ \sigma_2(\{2i-1, 2i\}) = \{2i-1, 2i\}$.
Let $\sigma = \sigma_2\sigma_1$. Permutations of such form make a subgroup of $S(\infty)$ which is isomorphic to $S(\infty)[Z_2]$. We will describe three important subgroups $S_n(\infty)[Z_2],\ \tilde{S}_n(\infty)[Z_2],\ S(n)[Z_2]$ of $S(\infty)[Z_2]$. Notice that the permutations of the type $\{\sigma_1\}$ and $\{\sigma_2\}$, where σ_1 and σ_2 are as described above, constitute subgroups of $S(\infty)[Z_2]$ which we will denote as Σ_1 and Σ_2. Then for any $\sigma_2\sigma_1 \in S(\infty)[Z_2]$,

$$\begin{aligned}
S_n(\infty)[Z_2] &= \{\sigma_2\sigma_1 \mid \sigma_2 \in \Sigma_2 \text{ and } \sigma_2(i) = i,\ i = 1, \cdots, 2n;\ \tilde{\sigma}_1 \in S_n(\infty)\}, \\
\tilde{S}_n(\infty)[Z_2] &= \{\sigma_2\sigma_1 \mid \sigma_2 \in \Sigma_2;\ \tilde{\sigma}_1 \in \tilde{S}_n(\infty)\}, \\
S(n)[Z_2] &= \{\sigma_2\sigma_1 \mid \tilde{\sigma}_1 \in S(n);\ \sigma_2(i) = i,\ i \geq 2n+1\}, \\
S^0(\infty)[Z_2] &= \{\sigma_2\sigma_1 \mid \tilde{\sigma}_1 \in S^0(\infty);\ \sigma_2(i) = i,\ i \geq 2N+1 \text{ for some } N\}.
\end{aligned}$$

Similarly to $S^0(\infty)$, we also can define $S^0(\infty)[Z_2]$ by

$$S^0(\infty)[Z_2] = \bigcup_{n=1}^{\infty} S(n)[Z_2].$$

$S^0(\infty)[Z_2]$ is the set of all the finite permutations of $S(\infty)[Z_2]$ and $S^0(\infty)[Z_2]$ is a countable normal subgroup in $S(\infty)[Z_2]$. We consider $S(\infty)[Z_2]$ with topology of pointwise convergence. The subgroups $S_n(\infty)[Z_2]$ make a fundamental family of neighborhoods of the identity.

4. UNITARY REPRESENTATIONS OF $S(\infty)[Z_2]$

4.1 Unitary representations of $S(\infty)[Z_2]$, Part I

Let π be an element of $\widehat{S(n)[Z_2]}$, the finite set of all equivalence classes of the irreducible unitary representations of $S(n)[Z_2]$. Consider π as a representation of $\tilde{S}_n(\infty)[Z_2]$ which is trivial on $S_n(\infty)[Z_2]$. Let $T = T(n,\pi)$ denote the unitary representation of $S(\infty)$ which is induced by π, which we now describe.

The space H_T is the space of all functions $h : S(\infty)[Z_2] \to H_\rho$ such that

$$h(\sigma_1\sigma) = \pi(\sigma_1)h(\sigma), \quad \sigma_1 \in \tilde{S}_n(\infty)[Z_2],\ \sigma \in S(\infty)[Z_2].$$

The norm in H_ρ is defined by

$$\|h\|^2 = \sum_{\dot{\sigma}\in\Lambda} \|h(\dot{\sigma})\|^2_{H_\rho} < \infty,$$

where $\dot{\sigma} = \{\sigma_1\sigma \mid \sigma \in S(\infty)[Z_2],\ \sigma_1 \in \tilde{S}_n(\infty)[Z_2]\}$ and $\Lambda = \tilde{S}_n(\infty)[Z_2] \setminus S(\infty)[Z_2]$ is the corresponding partition of $S(\infty)[Z_2]$. Note that $\|h(\cdot)\|$ is constant on any class modulo $\tilde{S}_n(\infty)[Z_2]$.

Then $S(\infty)[Z_2]$ acts in H_T as follows:

$$T(\sigma)h(\sigma') = h(\sigma'\sigma) \quad \text{for every } \sigma, \sigma' \in S(\infty)[Z_2].$$

There is another description of $T(n,\pi)$. Let $M = M(n)$ denote the set of all injective maps

$$M = M(n) = \{\mu \mid \mu : \{(i, i+1)\} \to \mu(\mu(i), \mu(i)+1),\ 1 \le i \le 2n-1\}.$$

Any $\mu = ((i_1, i_2), \cdots, (i_{2n-1}, i_{2n})) \in M$ can be considered an ordered sequence of distinct pairs. If $\mu = ((i_1, i_2), \cdots, (i_{2n-1}, i_{2n}))$ and $\sigma \in S(\infty)[Z_2]$, let $\mu\sigma = ((i_1, i_2)\sigma, \cdots, (i_{2n-1}, i_{2n})\sigma)$. And also let

$$\sigma'\mu = ((i_{\sigma'(1)}, i_{\sigma'(2)}), \cdots, (i_{\sigma'(2n-1)}, i_{\sigma'(2n)})), \quad \sigma' \in S(n)[Z_2].$$

So $S(n)[Z_2]$ acts on M on the left and $S(\infty)[Z_2]$ on the right.

The space H_T can be considered as the space of all the functions $h : M \to H_\pi$ such that

$$h(\sigma'\mu) = \pi(\sigma')h(\mu) \quad \text{for every } \sigma' \in S(\infty)[Z_2],\ \mu \in M.$$

The norm in H_ρ is defined by

$$\|h\|^2 = \sum_{\mu \in M} \|h(\mu)\|^2_{H_\rho} < \infty\ .$$

Then $S(\infty)[Z_2]$ acts in H_T as follows:

$$T(\sigma)h(\mu) = h(\mu\sigma) \quad \text{for every } \mu \in M \text{ and } \sigma \in S(\infty)[Z_2].$$

Let us define the subspace of all $S_m(\infty)[Z_2]$-invariant vectors in H_T as

$$H_T^m = \{h \in H_T \ \mid T(\sigma)h(\mu) = h(\mu),\ \sigma \in S_m(\infty)[Z_2],\ \text{ for any } m = 1, 2, \cdots,\ \mu \in M\}.$$

Let also us define the projection in H_T as

$$P_m : H_T \to H_T^m.$$

For any $m \geq n$ we set

$$M_m = M_m(n) = \{\mu \in M \mid \mu = ((i_1, i_2), \cdots, (i_{2n-1}, i_{2n})) \subseteq \{(1,2), \cdots, (2m-1, 2m)\}\ \},$$

$$H_T^{m'} = \{h \in H_T \mid supp(h) \subseteq M_m\}.$$

4.2 Unitary representations of $S(\infty)[Z_2]$, Part II

Lemma 4.1. *Let* $T = T(n, \pi)$, *where* $n = 1, 2, \cdots,\ \pi \in \widehat{S(n)[Z_2]}$.

$$H_T^m = \begin{cases} \{0\} & \text{if } m < n, \\ H_T^{m'} & \text{if } m \geq n. \end{cases}$$

Proof: Recall that $H_T^m = \{h \in H_T \mid T(\sigma_1)h(\mu) = h(\mu),\ \sigma_1 \in S_m(\infty)[Z_2],\ \mu \in M\}$. Since $m < n$, there is no such vectors $h(\mu)$ such that $T(\sigma_1)h(\mu) = h(\mu)$ and $\sigma_1 \in S_m(\infty)[Z_2]$. So $H_T^m = \{0\}$ if $m < n$. Since $S_m(\infty)[Z_2]$ acts on $M_m = \{\ \mu \in M \mid \mu = ((i_1, i_2), \cdots, (i_{2n-1}, i_{2n})) \subseteq \{(1,2), \cdots, (2m-1, 2m)\}\ \}$ trivially, so $H_T^{m'} \subseteq H_T^m$. Conversely, let $h \in H_T^m$. Then the function $\mu \mapsto \|h(\mu)\|$ is constant on $S_m(\infty)[Z_2]$-orbits in M. Since the sum of $\|h(\mu)\|^2$ over M is finite it follows that $\|h(\mu)\| = 0$ when the orbit containing μ is infinite. But this is exactly the case when $\mu \nsubseteq \{(1,2), \cdots, (2m-1, 2m)\}$. So $H_T^m \subseteq H_T^{m'}$. Thus $H_T^m = H_T^{m'}$. $\Diamond$

Lemma 4.2. *If* $m \geq n$ *and* $h \in H_T$, *then* $P_m h = \chi_m h$ *where* χ_m *is the characteristic function of the subset* $M_m \subset M$.

Proof: Let $m \geq n$, $h \in H_T$. After we multiply by χ_m where χ_m is a projection in H_T we obtain $P_m h = \chi_m h$. $\Diamond$

Lemma 4.3. *For any* $n = 1, 2, \cdots$ *and* $\pi \in \widehat{S(n)[Z_2]}$, $T = T(n, \pi)$ *is irreducible and the representations* $T(n, \pi)$ *are pairwise inequivalent.*

Proof: The space $H_T^{n'}$ is $S(n)[Z_2]$-irreducible and cyclic in H_T. Since $H_T^n = H_T^{n'}$, H_T^n is also $S(n)[Z_2]$-irreducible and cyclic in H_T. Let $H_T = H \oplus H^\perp$ be an orthogonal decomposition into invariant subspaces. Then

$$H_T^n = (H_T^n \cap H) \oplus (H_T^n \cap H^\perp).$$

Since H_T^n is $S(n)$-irreducible we have $H_T^n \subseteq H$ or $H_T^n \subseteq H^\perp$. But H_T^n is cyclic, so either $H = H_T$ or $H^\perp = H_T$. So $T = (n, \pi)$ is irreducible.
Since $n = min\{m \mid H_{T(n,\pi)}^m \neq \{0\}\}$, (n, π) is an invariant of $T(n, \pi)$. And π is the representation of $S(n)[Z_2]$ in $H_{T(n,\pi)}^m$. So the representations $T(n, \pi)$ are pairwise inequivalent. $\Diamond$

Lemma 4.4. *Any representation $T = T(n, \pi)$ is continuous in the topology we introduced on $S(\infty)[Z_2]$.*

Proof: It is sufficient to verify the continuity at $E \in S(\infty)[Z_2]$. For fixed $h \in H_T$, we have to prove that $T(\sigma)h \to h$ as $\sigma \in S(\infty)[Z_2]$ is close to E. Let $h_i = P_i h = \chi_i h$. Fix $\epsilon > 0$, then we can find i such that $\|h - h_i\| \leq \frac{\epsilon}{2}$. Now suppose that $\sigma \in S_i(\infty)[Z_2]$. Then $T(\sigma)h_i = h_i$. Therefore

$$\begin{aligned} \|T(\sigma)h - h\| &= \|T(\sigma)(h - h_i) - (h - h_i)\| \\ &\leq \|T(\sigma)(h - h_i)\| + \|h - h_i\| \\ &= 2\|h - h_i\| \\ &\leq \epsilon. \Diamond \end{aligned}$$

4.3 Tame representations of $S(\infty)[Z_2]$ and semigroups

In terms of matrices, the map

$$\alpha_n : \Phi(\infty)[Z_2] \to \Phi(n)[Z_2]$$

is defined as follows: if $\phi \in \Phi(\infty)[Z_2]$ then $\alpha_n(\phi)$ is simply the upper left $2n \times 2n$ sub-matrix of the matrix ϕ. Because $S^0(\infty)[Z_2] \subset S(\infty)[Z_2] \subset \Phi(\infty)[Z_2]$, for fix a number $n = 1, 2, \cdots$ we see that the map α_n is defined on $S^0(\infty)[Z_2]$.

Lemma 4.5. *(a) The mapping $\alpha_n : S^0(\infty)[Z_2] \to \Phi(n)[Z_2]$ is a surjection.*

(b) Let $\sigma_1,\ \sigma_2 \in S^0(\infty)[Z_2]$. Then $\alpha_n(\sigma_1) = \alpha_n(\sigma_2)$ if and only if σ_1 and σ_2 are in the same double class modulo $S^0_n(\infty)[Z_2]$ where $S^0_n(\infty)[Z_2] = S_n(\infty)[Z_2] \cap S^0(\infty)[Z_2]$.

Proof (a): Let $1_k,\ 0_k$ be the diagonal matrix of order k with 1's and 0's on the diagonal respectively. Let $\phi \in \Phi(n)[Z_2]$ be given. We have to find $\sigma \in S^0(\infty)[Z_2]$ such that $\alpha_n(\sigma) = \phi$. Suppose $\phi = \beta_k$ where $0 \le k \le 2n$ and

$$\beta_k = \begin{bmatrix} 1_{2k} & 0 \\ 0 & 0_{2n-2k} \end{bmatrix}.$$

Then we may set $\sigma = \sigma_{(k)}$ where

$$\sigma_{(k)} = \begin{bmatrix} 1_{2k} & 0 & 0 & 0 \\ 0 & 0_{2n-2k} & 1_{2n-2k} & 0 \\ 0 & 1_{2n-2k} & 0_{2n-2k} & 0 \\ 0 & 0 & 0 & 1_{2k} \end{bmatrix}.$$

Now let $\phi \in \Phi(n)[Z_2]$ be arbitrary and k denote the number of 1's in ϕ. Then we can find $\sigma',\ \sigma'' \in S(n)[Z_2] \subset S^0(\infty)[Z_2]$ such that $\phi = \sigma'\beta_k\sigma''$. But then we set

$\sigma = \sigma' \sigma_{(k)} \sigma''$. $\Diamond$

Proof: (b) ($\Leftarrow$) Suppose $\alpha_n(\sigma) = \beta_k$ where $\sigma \in S^0(\infty)[Z_2]$. Using some elementary calculations and the condition which in any row and any column the number of 1's is exactly one, we can easily verify that σ must lie in the same double class as $\sigma_{(k)}$. Now let $\alpha_n(\sigma_1) = \alpha_n(\sigma_2)$. Without the loss of generality we can $\alpha_n(\sigma_1) = \alpha_n(\sigma_2) = \beta_k$. But then σ_1 and σ_2 are in the same double class as $\sigma_{(k)}$.
($\Rightarrow$) If σ_1 and σ_2 are in the same double class then obviously $\alpha_n(\sigma_1) = \alpha_n(\sigma_2)$. $\Diamond$

Corollary 1. *$S^0(\infty)[Z_2]$ (and consequently $S(\infty)[Z_2]$) is dense in $\Phi(\infty)[Z_2]$.*

Let T be a continuous unitary representation of $S(\infty)[Z_2]$ and we denote H_T^n by the subspace of all $S_n(\infty)[Z_2]$-invariant vectors in H_T and set $H_T^\infty = \bigcup_{n=0}^{\infty} H_T^n$. Similarly, if T^0 be a continuous unitary representation of $S^0(\infty)[Z_2]$ and we denote $H_{T^0}^n$ by the subspace of all $S_n^0(\infty)[Z_2]$-invariant vectors in H_{T^0} and set $H_{T^0}^\infty = \bigcup_{n=0}^{\infty} H_{T^0}^n$.

Definition 4.1. *A unitary representation T^0 of $S^0(\infty)[Z_2]$ is said to be tame if $H_{T^0}^\infty$ is dense in H_{T^0}.*

Theorem 4.1. *Let T^0 be a unitary representation of $S^0(\infty)[Z_2]$. Then the following conditions are equivalent:*

(a) T^0 is tame.

(b) We can extend T^0 to a continuous unitary representation T of $S(\infty)[Z_2]$ which acts in the same space.

(c) We can extend T^0 to a representation ρ of the semigroup $\Phi(\infty)[Z_2]$ which acts in the same space.

Proof: $(c) \Rightarrow (b)$: Let $G = \{\phi \in \Phi(\infty)[Z_2] \mid \phi^*\phi = \phi\phi^* = 1\}$ be a group. $\rho|_G$ is a unitary representation of G. And if $\sigma \in S$, then $\rho(\sigma)^* = \rho(\sigma^*) = \rho(\sigma^{-1}) = \rho(\sigma)^{-1}$ so $\rho(\sigma)$ is unitary. $S(\infty)[Z_2]$ has a topology so that $\{S_n(\infty)[Z_2]\}_{n\geq 1}$ is a fundamental family of neighborhoods of the identity.$\diamondsuit$

Proof: $(b) \Rightarrow (a)$: Let $\psi \in H_T$ and $\epsilon > 0$ be arbitrary. Since T is continuous, we can find some m such that

$$\|T(\sigma)\psi - \psi\| \leq \epsilon \text{ for any } \sigma \in S_n(\infty)[Z_2].$$

By Birkhoff's Ergodic Theorem (see Appendix B), there exists ψ_0 such that $\|\psi - \psi_0\| \leq \epsilon$, and ψ_0 is $S_n(\infty)[Z_2]$-invariant. So H_T^∞ is dense in H_T.$\diamondsuit$

Proof: $(a) \Rightarrow (c)$: Let $P_n : H_{T^0} \rightarrow H_{T^0}^n$ be the orthogonal projection. Suppose n to be so large that $H_{T^0}^n \neq \{0\}$. By Lemma (4.5) (b), for any $\sigma_1, \sigma_2 \in S^0(\infty)[Z_2]$, we have

$$\alpha_n(\sigma_1) = \alpha_n(\sigma_2) \Longrightarrow P_n T^0(\sigma_1) P_n = P_n T^0(\sigma_2) P_n.$$

Since $\alpha_n(S^0(\infty)[Z_2]) = \Phi(n)[Z_2]$ and Lemma (4.5) (a), it follows that there exists a unique map

$$\rho_n : \Phi(n)[Z_2] \rightarrow \Phi(H_{T^0}^n)$$

such that $\rho_n(\alpha_n(\sigma)) = P_n T^0(\sigma)|_{H_{T^0}^n}$ for every $\sigma \in S^0(\infty)[Z_2]$.

For any $m > n$, there is a map

$$\alpha_{n,m} : \Phi(\infty)[Z_2] \rightarrow \Phi(n)[Z_2],$$

such that $\alpha_{n,m} \circ \alpha_n = \alpha_n$ where $\alpha_n : \Phi(\infty)[Z_2] \rightarrow \Phi(n)[Z_2]$.

Remark that $\rho_n(\alpha_{n,m}(\phi)) = P_n \rho_m(\sigma)|_{H_{T^0}^n}$ for every $\phi \in \Phi(m)[Z_2]$. It follows that for any $\phi \in \Phi(\infty)[Z_2]$ there exists an operator $\rho(\phi) \in \Phi(H_T)$ such that

$$P_n\rho(\phi)|_{H^n_{T0}} = \rho_n(\alpha_n(\phi)) \text{ for every } n.$$

Since $H^\infty_{T^0}$ is dense in H_{T^0}, this construction and the definition of topology in $\Phi(\infty)[Z_2]$ implies that $\rho : \Phi(\infty)[Z_2] \to \Phi(H_{T^0})$ is continuous. And ρ commutes with the involution and $\rho(1) = 1$.

Now, we have to show that

$$\rho(\phi_1\phi_2) = \rho(\phi_1)\rho(\phi_2) \text{ for every } \phi_1, \phi_2 \in \Phi(\infty)[Z_2]. \tag{4.1}$$

Let $\phi_1 \in S^0(\infty)[Z_2]$ be fixed. Since the multiplication in $\Phi(\infty)[Z_2]$ is separately continuous, (4.1) is continuous in ϕ_2. But $S^0(\infty)[Z_2]$ is dense in $\Phi(\infty)[Z_2]$. So (4.1) holds for any $\phi_2 \in \Phi_2[Z_2]$. We obtain for ϕ_1 by same reasoning. $\Diamond$

From Theorem (4.1), we can deduce the following sets:

1. the set of tame representations of $S^0(\infty)[Z_2]$,
2. the set of continuous unitary representations of $S(\infty)[Z_2]$,
3. the set of representations of $\Phi(\infty)[Z_2]$

are the same. Notice that the representations T and ρ in Theorem (4.1) are unique since $S^0(\infty)[Z_2]$ is dense in $S(\infty)[Z_2]$ and $\Phi(\infty)[Z_2]$.

Theorem 4.2. *Let T be a tame representation of $S^0(\infty)[Z_2]$ and $H^n_T \neq \{0\}$. Then the map $\rho_n : \Phi(n)[Z_2] \to \Phi(H^n_T)$ is a representation of $\Phi(n)[Z_2]$ in H^n_T.*

Proof: Let ρ be the representation of $\Phi(\infty)[Z_2]$. Consider the infinite matrix

$$\beta_{2n} = \begin{bmatrix} 1_{2n} & 0 \\ 0 & 0 \end{bmatrix} \in \Phi(\infty)[Z_2].$$

We claim that $\rho(\beta_{2n}) = P_n$ where $P_n : H_T \to H_T^n$ is orthogonal projection. We need to show that $H_T^{n'} = H_T^n$. First since $\beta_{2n} = \beta_{2n}^* = \beta_{2n}^2$, $\rho(\beta_{2n})$ is a projection. Let $H^{n'} = \rho(\beta_{2n})H_T$. For any $\sigma \in S_n^0(\infty)[Z_2]$, we have $\rho(\sigma)\rho(\beta_{2n}) = \rho(\beta_{2n})$ since $\sigma\beta_{2n} = \beta_{2n}$. It follows that $H^{n'} \subseteq H_T^n$.

Secondly, by the construction of ρ and ρ_n, we have

$$P_n\rho(\beta_{2n})|_{H_T^n} = \rho_n(\alpha_n(\beta_{2n})) = \rho_n(\alpha_n(1)) = P_nT(1)|_{H_T^n} = 1|_{H_T^n}.$$

Thus $H_T^n \subseteq H^{n'}$. So $H^{n'} = H_T^n$ and $\rho(\beta_{2n}) = P_n$.

Now to apply this, we try to prove that ρ_n is multiplicative. Consider $L(H_T^n)$ as a subalgebra of $L(H_T)$ by embedding $A \mapsto P_nAP_n$. Consider a multiplicative embedding $\Phi(n)[Z_2] \to \Phi(\infty)[Z_2]$ given by

$$\phi \mapsto \hat{\phi} = \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for every } \phi \in \Phi(n)[Z_2].$$

Then

$$\rho_n(\phi) = P_n\rho_n(\phi)P_n = P_n\rho(\hat{\phi})P_n = \rho(\beta_{2n})\rho(\hat{\phi})\rho(\beta_{2n}) = \rho(\beta_{2n}\hat{\phi}\beta_{2n}) = \rho(\hat{\phi}).$$

This yields the multiplicativity of ρ_n. Notice that $\rho_n\rho_n^* = \rho_n^*\rho_n$ with $\rho_n(1) = 1$. Thus ρ_n is a representation. $\Diamond$

For any $k > n$, let

$$\beta_{2n,k} = \begin{bmatrix} 1_{2n} & 0 \\ 0 & 0 \end{bmatrix} \in \Phi(k)[Z_2].$$

Lemma 4.6. *Let T be a tame representation of $S^0(\infty)[Z_2]$. Then $\rho_k(\beta_{2n,k})$ is the orthogonal projection from $H_T^k = H_{\rho_k}$ onto H_T^n.*

Proof: We extend T to a representation ρ of $\Phi(\infty)$ and use the fact $P_n = \rho(\beta_{2n})$. Since $\beta_{2n,k} = \alpha_k(\beta_{2n})$, we obtain

$$\rho_k(\beta_{2n,k}) = P_k\rho(\beta_{2n})|_{H_T^k} = P_kP_n|_{H_T^k} = P_n|_{H_T^k}.\Diamond$$

4.4 Representations of $S(\infty)[Z_2]$

The results of previous section can be used to study of the continuous representations of $S(\infty)[Z_2]$ by looking at continuous representations of the finite semigroups $\Phi(m)[Z_2],\ m = 1, 2, \cdots$. We need some facts about $\Phi(m)[Z_2]$.

Definition 4.2. *An element $\beta \in \Phi(m)[Z_2]$ is said to be idempotent if $\beta = \beta^* = \beta^2$.*

Any idempotent element has a form β_X where X is a finite subset of $\{(1, 2), \cdots, (2m - 1, 2m)\}$ with the domain $D(\beta_X) = X$ and the image $I(\beta_X) = X$. Then $\beta_X : X \to X$ is identity map. Notice that that $\beta_\varnothing = 0$ and $\beta_{\{(1,2),\cdots,(2m-1,2m)\}} = 1$. There is a natural partial ordering in the set of idempotent elements, $\beta_X \le \beta_Y$ if $X \subseteq Y$. For any $n = 0, 1, \cdots, m$, let

$$\beta_{2n} = \begin{cases} \beta_{\{(1,2),\cdots,(2n-1,2n)\}} & \text{if } n \neq 0, \\ 0 \text{ if } & \text{otherwise} . \end{cases}$$

Then any idempotent element $\beta \in \Phi(m)[Z_2]$ can be written as $\sigma\beta_{2n}\sigma^{-1}$ where $n = Card(D(\beta))$ and $\sigma \in S(m)[Z_2]$.

Let ρ be a representation of $\Phi(m)[Z_2]$. Then $\rho(\beta_X) = \rho(\beta_X)^* = \rho(\beta_X)^2$ hence $\rho(\beta_X)$ is a projection for $X \subseteq \{(1, 2), \cdots, (2m - 1, 2m)\}$. Let H_ρ^X be its range. Then $H_\rho^X \subseteq H_\rho^Y$ if

$X \subseteq Y$. Let $n(\rho)$ be the least number n such that $H_\rho^n = \{0\}$. For simple notation for H_ρ^n:

$$H_\rho^n = \begin{cases} H_\rho^{\{1,2,\cdots,2n\}} & \text{if } n \neq 0, \\ H_\rho^{\varnothing} & \text{if } n = 0. \end{cases}$$

Let

$$H_\rho^n = \begin{cases} \rho(\beta_X) = 0 & \text{if } Card(X) < n(\rho), \\ \rho(\beta_X) \neq 0 & \text{if } Card(X) \geq n(\rho). \end{cases}$$

Lemma 4.7. *Let ρ be a representation of $\Phi(m)[Z_2]$, ϕ an element of $\Phi(m)[Z_2]$ and $|\phi| = Card(D(\phi))$. Then*

$$\rho(\phi) = \begin{cases} \neq 0 & \textit{if } |\phi| \geq n(\rho), \\ = 0 & \textit{if } |\phi| < n(\rho). \end{cases}$$

Proof: Let $X = D(\phi)$. Then $\phi^*\phi = \beta_X$. We have

$$\begin{aligned} \rho(\phi) \neq 0 &\iff \rho(\phi)^*\rho(\phi) \neq 0 \\ &\iff \rho(\phi^*\phi) \neq 0 \\ &\iff \rho(\beta_X) \neq 0 \end{aligned}$$

and the lemma follows. $\Diamond$

Lemma 4.8. *Let ρ be an irreducible representation of $\Phi(m)[Z_2]$ and $n = n(\rho) \neq 0$. Then H_ρ^n is invariant and irreducible under the action of $S(n)[Z_2]$.*

Proof: Since $\Phi(m)[Z_2]$ is finite, ρ is finite-dimensional. Since a representation ρ is irreducible, the operator $\rho(\phi),\ \phi \in \Phi(m)[Z_2]$, spans $L(H(\rho))$. So H_ρ^n is $S(n)[Z_2]$-invariant. For irreducibility of H_ρ^n under $S(n)[Z_2]$, it is sufficient to show that the operator $\rho(\sigma)|_{H_\rho^n},\ \sigma \in S(n)[Z_2]$ spans $L(H_\rho^n)$.

Remark that the operators $\rho(\beta_{2n})\rho(\phi)\rho(\beta_{2n})|_{H_\rho^n},\ \phi \in \Phi(m)[Z_2]$, span $L(H_\rho^n)$. Let

$\phi' = \beta_{2n}\phi\beta_{2n}$. Then $\rho(\beta_{2n})\rho(\phi)\rho(\beta_{2n}) = \rho(\phi')$. By Lemma (4.7) we have $\rho(\phi') = 0$ unless $|\phi'| = n$. But $|\phi'| = n \Longleftrightarrow \alpha_n(\phi') \in \alpha_n(S(n)[Z_2])$. $\Diamond$

Lemma 4.9. *Let ρ be an irreducible representation of $\Phi(m)[Z_2]$ and $n = n(\rho) \neq 0$. Let $\pi \in \widehat{S(n)[Z_2]}$ be the representation of $S(n)[Z_2]$ in H^n_ρ. Then ρ is uniquely determined by (n, π).*

Proof: For any $\psi \in H_\pi = H^n_\rho$ and any $\phi \in \Phi(m)[Z_2]$ we have

$$\begin{aligned}
(\rho(\phi)\psi, \psi) &= (\rho(\phi)\rho(\beta_{2n})\psi, \rho(\beta_{2n})\psi) \\
&= (\rho(\beta_{2n}\phi\beta_{2n}), \psi) \\
&= \begin{cases} (\pi(\phi)\psi, \psi) \text{ if } \alpha_n(\beta_{2n}\phi\beta_{2n}) = \alpha_n(\sigma) \text{ where } \sigma \in S(n)[Z_2], \\ 0 \text{ otherwise .} \end{cases}
\end{aligned}$$

Since any nonzero $\psi \in H^n_T$ is cyclic, it is sufficient to apply Lemma (2.4). So ρ and π are equivalent. $\Diamond$

For any pair (n, π) where $n = 1, 2, \cdots, m$ and $\pi \in \widehat{S(n)[Z_2]}$, we construct an irreducible representation $\rho = \rho(n, m; \pi)$ such that $n = n(\rho)$ and π is the representation of $S(n)[Z_2]$ in H^n_ρ. Recall $M_m(n)$, the set of all injective maps on pairs:

$$\mu : \{(1, 2), \cdots, (2n-1, 2n)\} \to \{(1, 2), \cdots, (2m-1, 2m)\}$$

The space H_ρ is the space of all functions $h : M_m(n) \to H_\rho$ such that

$$h(\sigma'\mu) = \rho(\sigma')h(\mu) \text{ every } \mu \in M(n),\ \sigma' \in S(n)[Z_2].$$

The norm in H_ρ is defined by

$$\|h\|^2 = \sum_{\mu \in M} \|h(\mu)\|^2_{H_\pi}$$

We set $\mu\phi = ((i_1, i_2)\phi, \cdots, (i_{2n-1}, i_{2n})\phi)$. Then $\Phi(m)[Z_2]$ acts in H_ρ as follows

$$\rho(\phi)h(\mu) = \begin{cases} h(\mu\phi) \text{ if } \mu = ((i_1, i_2), \cdots, (i_{2n-1}, i_{2n})) \subseteq D(\phi), \\ 0 \text{ otherwise } |\phi| < n(\rho). \end{cases}$$

One can easily check that ρ is a representation of $\Phi(m)[Z_2]$.

For any $X \subseteq \{(1,2), \cdots, (2m-1, 2m)\}$, H_ρ^X consists of those h which are concentrated on $\{\mu \mid \mu \subseteq X\}$. It follows that $n(\rho) = n$ and that the representation of $S(n)[Z_2]$ in H_ρ^n is π. H_ρ^n is cyclic under $\Phi(m)[Z_2]$, the irreducibility of ρ can be proved in exactly same way by Lemma (4.3). $\Diamond$

Theorem 4.3. *Every irreducible representation of* $\Phi(m)[Z_2]$ *is either trivial or one of the representations* $\rho(n, m; \pi)$ *where* $n = 1, 2, \cdots, m$ *and* $\pi \in \widehat{S(n)[Z_2]}$.

Proof: Let ρ be an irreducible representation of $\Phi(m)[Z_2]$. If $n = n(\rho) \neq 0$ then ρ is equivalent to some $\rho(n, m; \pi)$ by Lemma (4.9). But if $n = n(\rho) = 0$ then $\rho(0) \neq 0$ and so ρ is trivial by Lemma (2.5).

Let $T = T(n, \rho)$ and $m \geq n$. Consider the representation ρ_m of $\Phi(m)[Z_2]$ in the space H_T^m which is afforded by Theorem (4.2). Then ρ_m is equivalent to $\rho(n, m; \pi)$. To see this we use Lemma (4.2) and compare the definition of $\rho(n, m; \pi)$ with that of $T(n, \pi)$. $\Diamond$

5. THE EXTENSION OF LIEBERMAN'S THEOREM FOR $S(\infty)[Z_2]$

Theorem 5.1. *(a) Let T be an irreducible continuous unitary representation of $S(\infty)[Z_2]$ and let $\{T(n,\pi), n = 1, 2, \cdots, \pi \in \widehat{S(n)[Z_2]}\}$ be a continuous irreducible unitary representation of $S(\infty)[Z_2]$. Then there exist $n,\ \pi$ such that $T(n,\pi)$ is equivalent to T.*

(b) Any continuous unitary representation of $S(\infty)[Z_2]$ has an irreducible subrepresentation.

(c) Any continuous unitary representation of $S(\infty)[Z_2]$ can be decomposed into a direct sum of irreducible representations.

Proof: (a) Let T be an irreducible continuous unitary representation of $S(\infty)[Z_2]$. (It means that T be an irreducible tame representation of $S^0(\infty)[Z_2]$). Let $n(T)$ be the least number n such that $H_T^n \neq 0$. If $n(T) = 0$ then T is the trivial representation. Suppose $n = n(T) \neq 0$. We will prove that T is equivalent to some $T(n,\pi)$. For any $m \geq n$, the linear span of the set of operators

$$P_m T(\sigma)|_{H_T^m},\ \sigma \in S^0(\infty)[Z_2]$$

is weakly dense in $L(H_T^m)$. Therefore the linear span of the operators $\rho_m(\phi), \phi \in \Phi(m)[Z_2]$ is dense in $L(H_T^m)$. Hence ρ_m is irreducible. By Lemma (4.6), we have $n(\rho_m) = n$ for any $m \geq n$, whence $\rho_m = \rho(n,m;\pi)$ where π is the representation of $S(n)[Z_2]$ in H_T^m. Let $m = n$. By Lemma (2.2), for any $\phi \in \Phi(n)[Z_2]$ we have

$$\rho_n(\phi) = \begin{cases} \pi(\phi) \text{ if } \phi \in S(n)[Z_2], \\ 0 \text{ otherwise .} \end{cases}$$

Finally any $\psi \in H_T^m = H_\pi$ and any $\sigma \in S^0(\infty)[Z_2]$,

$$\begin{aligned}(T(\sigma)\psi,\psi) &= (P_nT(\sigma)P_n\psi,\psi) = (\rho_n(\alpha_n(\sigma))\psi,\psi) = \\ &= \begin{cases}(\pi(\alpha_n(\sigma))\psi,\psi) \text{ if } \alpha_n(\sigma) \in S(n)[Z_2], \\ 0 \text{ otherwise .}\end{cases}\end{aligned}$$

By Corollary (1), T is uniquely determined by (n,π) and therefore T is equivalent to $T(n,\pi)$. $\Diamond$

Proof: (b) Let T be a continuous unitary representation of $S(\infty)[Z_2]$ (T be a tame representation of $S^0(\infty)[Z_2]$.) We will prove that there exists an irreducible subrepresentation of T. Let n be so large that $H_T^n \neq 0$. Consider the representation ρ_n of the semigroup $\Phi(n)[Z_2]$ in H_T^n. Lemma (2.3) shows that there exists a subspace $H \subset H_\rho^n$ which is invariant and irreducible under $\Phi(n)[Z_2]$. Let H' denote its cyclic span under the action of $S^0(\infty)[Z_2]$ and let T' be the corresponding representation of $S^0(\infty)[Z_2]$ in H'. For any $\sigma \in S^0(\infty)[Z_2]$, we have

$$P_nT(\sigma)H = P_nT(\sigma)P_nH = \rho_n(\alpha_n(\sigma))H \subseteq H.$$

It follows that $H_{T'}^n = H$. Therefore H is both $\Phi(n)[Z_2]$-irreducible and $S^0(\infty)[Z_2]$-cyclic. It remains to apply Lemma (4.3) to see that T' is irreducible. $\Diamond$

Proof: (c) Let π be an arbitrary representation of $S(\infty)[Z_2]$. We single out a special collection of subrepresentations of π which are discretely decomposable into irreducibles. The collection $\mathcal{D}$ of all such discretely decomposable subrepresentations of π can be given a partial order as follows. Let T and T' be representations from $\mathcal{D}$. Then $T < T'$ if $H(T) \leq H(T')$ and the decomposition of T is a subrepresentation of T'. A linearly ordered set $\mathcal{L}$ of such representations has an upper sum; namely, just the direct sum of all distinct components of all representations from $\mathcal{L}$.

By Zorn's Lemma (see Appendix E), $\mathcal{D}$ must have a maximal element, say T_1. Then the representation T_2 of $S(\infty)[Z_2]$ given on the orthogonal complement of H_{T_1} cannot have an irreducible subrepresentation for otherwise we obtain a contradiction to the maximality of T_1. This contradicts the part (b). $\Diamond$

6. CONCLUSION OF THIS THESIS

In this thesis, we used semigroup method to prove that any representation of the wreath product of the infinite symmetric group $S(\infty)$ with Z_2 with appropriate topology generates a von Neumann algebra of type I. In fact, the von Neumann algebra $\mathcal{M}$ is a discrete sum of algebras $*$-isomorphic to the algebra $L(H)$ for some Hilbert space H. In the future, I hope to extend the results of this thesis to $S(\infty)[Z_n],\ n \in \mathbb{N}$ and maybe to $S(\infty)[G]$ where G is an arbitrary compact group. It would be also very interesting to extend the original Lieberman's proof (See Appendix A) to prove the result of thesis and its generalizations based on Chapter 2.4 and 2.5.

Bibliography

[2] J. L. Alperin and R. B. Bell, Groups And Representations , Graduate Texts in Mathematics, Springer, 1995.

[2] G. Birkhoff, An Ergodic Theorem for general semi-groups , Mathematics, VOL. 25, 625-627, 1939.

[3] R. P. Boyer, Representation theory of infinite-dimensional unitary groups, In: Representation Theory of Groups and Algebras, in: Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, 381391.

[4] R. P. Boyer, Character theory of infinite wreath products, Int. J. Math, Math. Sci. (2005), 1365-1379.

[5] J. Dixmier, C^* Algebra , North Holland, Amsterdam, 1982.

[6] R. C. Fabec, Fundamentals of Infinite Dimensional Representation Theory, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Volume 114, 2000.

[7] W. Fulton and J. Harris, Representation theory, Graduate Texts in Mathematics, Springer, 1991.

[8] W. Fulton, Young Tableaux: with applications to representation theory and geometry, London Mathematical Society Student Texts, Cambridge University Press, 1997.

[9] J. Faraut, Infinite Dimensional Spherical Analysis, Notes written by Sho Matsumoto, Kyushu University, 2007.

[10] T. Hirai, E. Hirai, A. Hora, Realizations of factor representations of finite type with emphasis on their characters for wreath products of compact groups with the infinite symmetric group, J. Math. Kyoto Univ., 46 (2006), no. 1, 75-106.

[11] T. Hirai, Construction of irreducible unitary representations of the infinite symmetric group S_∞, J. Math. Kyoto Univ., 31, 1991, 495-541.

[12] T. Hirai, E. Hirai, and A. Hora, Limits of characters of wreath products $S_n(T)$ of a compact group T with the symmetric groups and characters of $S_\infty(T)$, I. Nagoya Math. J.,193 (2009), 1-93.

[13] A. Hora and N. Obata, Quantum probability and spectral analysis of graphs, Theoretical and Mathematical Physics, Springer, 2007.

[14] A. Hora, Lecture note on introduction to asymptotic theory for representations and characters of symmetric groups, Wroclaw University in Poland, 2008.

[15] G. D. James, The representation theory of the symmetric group, Advanced Book Program, Addison-Wesley Pub. Co., 1981.

[16] S. Kerov, G. Olshanski, and A.M. Vershik. Harmonic analysis on the infinite symmetric group, Invent. math., 158 (2004), 551-642.

[17] A.A. Kirillov, Representations of the infinite dimensional unitary group, Dokl. Akad. Nauk SSSR 212 (1973), 288-290; English transl. in Soviet Math. Doklady 14 (1973), 1355-1358.

[18] S. Lang, Algebra , Graduate Texts in Mathematics, Springer, 2002.

[19] A. Lieberman, The structure of certain unitary representations of infinite symmetric groups, Trans. Amer. Math. Soc., 164 (1972), 189-198.

[20] F.J. Murray and J. von Neumann. On rings of operators. Ann. of Math., 37 (2) (1936), 116-229.

[21] F. J. Murray and J. von Neumann, On rings of operators, IV, Ann. Math., 44 (1943), 716-808.

[22] N. Obata, Certain unitary representations of the infinite symmetric group I, Nagoya Math. J., 105 (1987), 109-119.

[23] N. Obata, Certain unitary representations of the infinite symmetric group II, Nagoya Math. J., 106 (1987), 143-162.

[24] A. Okounkov, On representations of the infinite symmetric group, J. Math. Sci. (New York), 96 (5) (1999), 3550-3589. extslarXiv:math/9803037v1 [math.RT].

[25] G. I. Olshanski, New large groups of type I, J. Soviet Math., 18 (1982), 22-39.

[26] G. I. Olshanski, Unitary representations of the infinite symmetric group: a semigroup approach. In: Representations of Lie groups and Lie algebras (A.A. Kirillov, ed.). Budapest, Akad. Kiado, 1985, 181-198.

[27] G. I. Olshanski, On semigroups related to infinite-dimensional groups. In: Topics in representation theory, volume 2 of Adv. Soviet Math., Providence, RI, 1991, 67-101.

[28] G. I. Olshanski, Unitary representations of (G, K)-pairs that are connected with the infinite symmetric group $S(\infty)$, Algebra i Analiz, 1 (4):178-209, 1989. Translation in Leningrad Math. J. 1 (1990), no. 4, 983-1014.

[29] G. I. Olshanski, Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians. In: Topics in representation theory, volume 2 of Adv. Soviet Math., Providence, RI, 1991, 1-66.

[30] G. Olshanski, An introduction to harmonic analysis on the infinite symmetric group. In: Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001), volume 1815 of Lecture Notes in Math. Springer, Berlin, 2003. Cambridge Univ. Press, 127-160.

[31] L. Pontrjagin, Topological groups, Princeton Mathematical Series, Princeton University Press, 1958.

[32] G. Robinson, Representation theory of the symmetric group, University of Toronto Press, 1961.

[33] F. Riesz and B. Szokefalvi-Nagy, Functional analysis, Translation of Lecons d'analyse fonctionelle, Dover Publications, 1990.

[34] B. Sagan, The symmetric group, Graduate Texts in Mathematics, Springer, 2001.

[35] S. Sakai, $C*$-algebras and $W*$-algebras, Classics in Mathematics, Springer, 1971.

[36] J. P. Serre, Linear Representations of Finite Groups , Graduate Texts in Mathematics, Springer, 1977.

[37] S. Stratila and D. Voiculescu, Representations of AF-aglebra and of the group $U(\infty)$, Lecture Notes in Math., vol. 486, Springer-Verlag, Berlin, Heidelberg, and New York, 1975.

[38] E. Thoma, Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzahlbar unendlichen, symmetrischen Gruppe, Math. Z., 85 (1964), 40-61.

[39] A. M. Vershik and S. V. Kerov, Asymptotic theory of characters of the symmetric group, Funct. Anal. Appl., 15 (1981), 246-255.

[40] A. M. Vershik and S. V. Kerov, Characters and factor representations of the infinite symmetric group, Soviet Math. Dokl., 23 (1981), 389-392.

[41] A. M. Vershik and S. V. Kerov, The K-functor (Grothendieck group) of the infinite symmetric group, J. Soviet Math., 28 (1985), 549-568.

[42] S. Gill Williamson, Combinatorics for computer science, Computers and Math Series, Dover Books on Mathematics, 1985.

APPENDIX A: Original proof of Lieberman's Theorem [19] [1]

Let $G(S)$ denote either the group of regular or signed permutations on the set S. Lieberman treats only the case of regular permutations.

Definition 6.1. *$G(S)$ is given the topology of pointwise convergence on S so $G(S)$ becomes a non-locally compact Polish group. If Q is a subset of S, let $G(Q)$ denote the subgroup of $G(S)$ that consists of all elements $g \in G(Q)$ such that $gx = x$ for all $x \notin Q$.*

Note that the union of all the finite subgroups $G(T)$ where T is a finite subset of S is a dense subgroup of $G(S)$.

Theorem 6.1. *Let Γ be a continuous representation of $G(S)$ on the Hilbert space H. For any nonzero vector v in H, there exists a finite subset Z_v of S such that the restriction of Γ to $G(S \setminus Z_v)$ contains the trivial representation of $G(S \setminus Z_v)$.*

Proof : Without loss of generality, we assume that $v \in H$ is a unit vector.

Step 1: Given any unit vector v in H, there exists a finite subset $Z = Z_v$ of S such that

$$\mathbb{R}\langle \Gamma(g)v, v\rangle \geq 1/2, \text{ for all } g \in G(S \setminus Z_v).$$

We argue by contradiction so assume that for each finite subset T of S there is an element $g_T \in G(S \setminus T)$ such that

$$\mathbb{R}\langle \Gamma(g_T)v, v\rangle \leq 1/2$$

[1] This appendix contains detailed explanation of the original Lieberman's proof, composed by Dr. Robert Boyer, and is put here with his permission.

and $g_T x = x$ for all $x \in T$. By construction, we find that

$$\sup\{\mathbb{R}(\langle\Gamma(g_T)v, v\rangle : T \text{ finite subset of } S\} \leq 1/2.$$

On the other hand, since the support of g_T must lie inside $S \setminus T$, we find that the group identity e is a limit point of the set $\{g_T : T \text{ finite subset of } S\}$ or

$$\lim_{T\in\mathcal{D}} g_T = e,$$

by continuity in the topology of pointwise convergence on S and $\mathcal{D}$ is the directed set of all finite subsets of S. Therefore $\lim_{T\in\mathcal{D}}\langle\Gamma(g_T)v, v\rangle = \langle\Gamma(e)v, v\rangle = 1$. Contradiction.

Aside: We can probably avoid limits over directed sets by considering

$$\sup\{\mathbb{R}\langle\Gamma(g_T)v, v\rangle : T \text{ finite subset of } S\} = \langle v, v\rangle = 1.$$

Therefore, there is a finite subset Z_v of S, that depends on the choice of unit vector v, such that

$$\mathbb{R}\langle\Gamma(g)v, v)\rangle \geq 1/2, \quad g \in G(S \setminus Z_v).$$

We introduce further notation. For finite subset T of S, let P_T be the standard projection onto the subspace of invariant vectors for the subgroup $G(T)$ of $G(S)$:

$$P_T = \frac{1}{|G(T)|}\sum_{g\in G(T)}\Gamma(g).$$

Step 2: Let P be the projection on H given by $\inf\ P_T$ where T ranges over all finite subsets T of $S \setminus Z_v$. Then $P \neq 0$.

Recall that the collection of all orthogonal projections on a Hilbert space forms a complete lattice. The range of $P = \inf\ P_T$ is the intersection of all the ranges $\mathrm{Ran}(P_T)$.

By construction, P is indeed a projection. Consider the inner product $\langle P_T v, v\rangle$. It suffices to verify that it is uniformly bounded away from zero over all finite subsets of T of $S \setminus Z_v$; in fact, $\langle Pv, v\rangle \geq 1/2$ by Step 1. To see this, consider

$$\begin{aligned}
\langle P_T v, v\rangle &= \mathbb{R}\langle P_T v, v\rangle \\
&= \frac{1}{|G(T)|} \sum_{g \in G(T)} \mathbb{R}\langle \Gamma(g) v, v\rangle \\
&\geq \frac{1}{|G(T)|} \sum_{g \in G(T)} \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

Step 3: The range of P is fixed under the subgroup $G(S \setminus Z_v)$.

Let y be any nonzero vector from the range of P, let T be any finite subset of $S \setminus Z_v$, and $g \in G(T)$. Since $\mathrm{Ran}(P) \subset \mathrm{Ran}(P_T)$, we find that

$$\Gamma(g) y = y$$

By continuity in the topology of pointwise convergence, $\Gamma(p)y = y$ for all $p \in G(S \setminus Z_v)$; that is, $\Gamma | G(S \setminus Z_v)$ acts as the trivial representation on $\mathrm{Ran}(P)$.

Step 4: Any vector in the range of P is fixed by elements from the subgroup $G(S \setminus Z_v)$.

We re-state the last line of the proof of Step 3 for emphasis. $\diamondsuit$.

By a standard application of Zorn's lemma (See Appendix F), we obtain following corollary:

Corollary 2. *Let Γ be a continuous representation of $G(S)$. Then Γ is a direct sum of continuous irreducible representations.*

We review here in detail Lieberman's argument that Γ splits into an orthogonal direct sum. He cites two lemmas:

- Let G be a group, H a Hilbert space, and Γ a representation of G on H. Let J be a closed subspace of H, P be the projection of H onto J, and let $G_J = \{g \in G : \Gamma(g)J = J\}$. Assume that

  (a) There exists $X \subseteq G$ such that $H = \oplus\{\Gamma(x)J : x \in X\}$.

  (b) If $g \in G$, then $\Gamma(g)J = J$ or $\Gamma(g)J \perp J$.

  (c) $P \in \Gamma(G)''$.

  Then the mapping $T \mapsto TP$ is an algebraic $*$-isomorphism from $\Gamma(G)'$ onto $\Gamma(G)'|J$ and

$$\Gamma(G)'|J = \Gamma(G_J)'|J.$$

- Let H be a Hilbert space, G a group, and Γ a representation of G on H. Let Q be a subset of G, $J = \{v \in H : \Gamma(g)v = v \text{ if } g \in Q\}$, and let P be the projection of H onto J. Then $P \in \Gamma(G)$".

Now Lieberman assumes that Γ is a representation of $G(S)$. It suffices to show that Γ contains an irreducible subrepresentation. An application of Zorn's Lemma will then complete the proof.

Assume Γ acts on the Hilbert space H.

Notation: if $Q \subset S$, let $H_Q = \{v \in H : \Gamma(g)v = v, \forall g \in G(S \setminus Q)\}$.

We know that there exists a finite subset Z of S such that $H_Z \neq 0$ and $H_Q = 0$ if $Q \subset S$ and $|Q| < |Z|$. Now $G(Z)$ is a finite group. Consequently, $\Gamma(G(Z))|H_Z$ is a direct sum of irreducible representations of $G(Z)$.

Let Γ_0 be an irreducible subrepresentation of $\Gamma(G(Z))|H_Z$. Assume Γ_0 acts on H_{0Z}. Let H_0 be the closed subspace of H generated by $\Gamma(G)H_{0Z}$. By the above two lemmas, $\Gamma|H_0$ is irreducible. This ends his proof.

APPENDIX B: Garrett Birkhoff Ergodic Theorem for Semigroups - 1939

Birkhoff Version of the Ergodic Theorem

Let $\mathcal{S}$ be a semigroup of contractions on a Hilbert Space H. Then the means of the transforms of any element of H converge to a fix-point.

Birkhoff's notion of convergence is a generalization of the Moore-Smith theory of directed sets. Let $\{x_\alpha\}$ be any set of elements from a topological space X. Suppose a transitive relation $x_\alpha < x_\beta$ (read x_β is a successor of x_α) is given on this set. Say that $\{x_\alpha\}$ "converges" to a limit a if given any neighborhood $U(a)$ of a, every x_α must have a successor x_β, all of whose successors lie also in $U(a)$.

Let $\xi \in H$. We can order the means of the transforms $S\xi$, $S \in \mathcal{S}$ as follows. We say that one mean is a "successor" of another mean if and only if it is a mean of its transforms. In other words, if $\sum c_a S_a \xi$ is any mean of transforms of ξ, then the "successors" of this mean are

$$\sum_b c'_b S_b \left(\sum_a c_a S_a \xi \right)$$

Finally, we call ξ a fix-point if $S\xi = \xi$ for all $S \in \mathcal{S}$.

Olshanski applies this result for groups of unitary operators. In particular, let T be a unitary representation of the completed version of $S(\infty)$. Let $\epsilon > 0$ and $\xi \in H_T$ be given. Suppose that there exists an integer n such that

$$\|T(\sigma)\xi - \xi\| \leq \epsilon$$

for all $\sigma \in S_n(\infty)$. It is clear that any mean of $T(\sigma)\xi$ in the sense of Birkhoff still must satisfy

$$\left\| \sum_a c_a T(\sigma)\xi - \xi \right\| \leq \epsilon$$

since $\sum_a c_a = 1$ and $c_a \geq 0$. By his version of the Ergodic Theorem, these means converge to a fix-point ξ_0; that is,

$$T(\sigma)\xi_0 = \xi_0, \text{ for all } \sigma \in S_n(\infty)[Z_2].$$

Hence, for any vector $\xi \in H_T$ and $\epsilon > 0$, there exists an integer n and a vector ξ_0 such that $\|\xi - \xi_0\| \leq \epsilon$ and $T(\sigma)\xi_0 = \xi_0$ for all $\sigma \in S_n(\infty)[Z_2]$.

APPENDIX C: The calculation of the wreath product of $S(3)$ with Z_2

The action of the wreath product can be described as in the case of (e, f) for first matrix and similar matrices of 7 of them can be also found:

$$
\begin{aligned}
(e, f_1)(1,1) &= (e1, f_1(1)1) = (1,1) \\
(e, f_1)(1,2) &= (e1, f_1(1)2) = (1,2) \\
(e, f_1)(2,1) &= (e2, f_1(2)1) = (2,1) \\
(e, f_1)(2,2) &= (e2, f_1(2)2) = (2,2) \\
(e, f_1)(3,1) &= (e3, f_1(3)1) = (3,1) \\
(e, f_1)(3,2) &= (e3, f_1(3)2) = (3,2)
\end{aligned}
$$

$$
\left[\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\left[\begin{array}{cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right].
$$

Similarly, we find the following matrices for **the case of** $((1\ 2), f)$:

$$\left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{array}\right].$$

Similarly, we find the following matrices for **the case of** $((1\ 3), f)$:

$$\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right].$$

Similarly, we find the following matrices for **the case of** $((2\ 3), f)$:

$$\left[\begin{array}{cc|cc|cc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right] \left[\begin{array}{cc|cc|cc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc|cc} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right].$$

Similarly, we find the following matrices for **the case of** $((1\ 2\ 3), f)$:

$$\left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{array}\right].$$

Similarly, we find the following matrices for **the case of** $((1\ 3\ 2), f)$:

$$\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc|cc} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right]\left[\begin{array}{cc|cc|cc} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ \hline 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right].$$

The subgroup $S(3)[Z_2]$ is isomorphic to a subgroup of $GL(6,\mathbb{R})$ which has 48 elements.

APPENDIX D: Second Proof of Theorem (4.2)

In this appendix we follow Olshanski argument from [26]. We need to show that

$$\rho_n(\phi_1)\rho_n(\phi_2) = \rho_n(\phi_1\phi_2) \text{ for every } \phi_1, \phi_2 \in \Phi(n)[Z_2].$$

Choose $\sigma_1, \sigma_2 \in S^0(\infty)[Z_2]$ such that $\alpha_n(\phi_1) = \phi_1,\ \alpha_n(\phi_2) = \phi_2$. Then

$$\rho_n(\phi_1)\rho_n(\phi_2) = P_n T(\sigma_1) P_n T(\sigma_2)|_{H_T^n}.$$

For any $m > n$, we set $S_n(m)[Z_2] = S(m)[Z_2] \cup S_n(\infty)[Z_2]$. Note that $S_n(m)[Z_2] \cong S(m-n)[Z_2]$. Let

$$P_{n,m} = \frac{1}{(m-n)!} \sum_{\sigma \in S_n(m)[Z_2]} T(\sigma).$$

The operator $P_{n,m}$ is the projection onto the subspace of all $S_n(m)[Z_2]$-invariant vectors in H_T. Since $S_n^0(\infty)[Z_2] = \bigcup_{m \geq n} S_n(m)[Z_2]$, we have

$$\overset{s}{\lim_{m\to\infty}} P_{n,m} = P_n,$$

where $\overset{s}{\lim}$ denotes the limit is taken relative to the strong operator topology. It follows that

$$\begin{aligned}
\rho_n(\phi_1)\rho_n(\phi_2) &= \overset{s}{\lim_{m\to\infty}} P_n T(\sigma) P_{n,m} T(\sigma_2)|_{H_T^n} \\
&= \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2]} P_n T(\sigma_1\sigma\sigma_2)|_{H_T^n} \\
&= \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2]} \rho_n(\alpha_n(\sigma_1\sigma\sigma_2)).
\end{aligned}$$

Fix a number $k > n$ and $\sigma_1, \sigma_2 \in S(k)[Z_2]$. Suppose that $m \geq k$ and write

$$\sigma_1 = \begin{bmatrix} \phi_1 & \alpha & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1_{2n} & 0 & 0 \\ 0 & x_\sigma & * \\ 0 & * & * \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \phi_2 & * & 0 \\ \beta & * & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where α is a $2n\times(2k-2n)$ matrix, x_σ is a $(2k-2n)\times(2k-2n)$ matrix, β is a $(2k-2n)\times 2n$ matrix. Then

$$\alpha_n(\sigma_1\sigma\sigma_2) = \phi_1\phi_2 + \alpha x_\sigma \beta.$$

Let $m' = m - n$ and $k' = k - n$. Identify $S_n(m)[Z_2] \subset \overline{S}_n(\infty)[Z_2]$ with $S(m')[Z_2] \subset S^0(\infty)[Z_2]$. Then x_σ can be considered as $\alpha_{k'}(\sigma)$. We claim that

$$\lim_{m'\to\infty} \frac{Card\{\ \sigma \in S(m')[Z_2] \mid \alpha_{k'}(\sigma) \neq 0\ \}}{(m')!} = 0.$$

It can be verified by using probability techniques. Consider the uniform probability measure on the group $S(m')[Z_2]$. Note that $\alpha_{k'}(\sigma) \neq 0 \Longleftrightarrow \sigma(i) \leq k'$ for some $i = 1, \cdots, k'$. But for given $i,\ 1 \leq i \leq k'$, the probability of the event $\sigma(i) \leq k'$ tends to zero when

$m' \to \infty$. So we obtain

$$\begin{aligned}
\rho_n(\phi_1)\rho_n(\phi_2) &= \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2], x_\sigma\neq 0} \rho_n(\alpha_n(\sigma_1\sigma\sigma_2)) \\
&+ \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2], x_\sigma = 0} \rho_n(\alpha_n(\sigma_1\sigma\sigma_2)) \\
&= 0 + \rho_n(\phi_1\phi_2)
\end{aligned}$$

since $\|\rho_n(\phi)\| \leq 1$ for any $\phi \in \Phi(n)[Z_2]$, the first term is zero and the second term is $\rho_n(\phi_1\phi_2)$. $\diamondsuit$

APPENDIX E: Second Proof of Lemma (4.6)

In this appendix we follow Olshanski argument from [26]. We begin the formula $P_n = \overset{s}{\lim\limits_{m\to\infty}} P_{n,m}$. Since $P_n = P_k P_n$ we have

$$\begin{aligned} P_n|_{H_T^k} &= \overset{s}{\lim_{m\to\infty}} P_k P_{n,m}|_{H_T^k} \\ &= \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2]} P_k T(\sigma)|_{H_T^k} \\ &= \overset{s}{\lim_{m\to\infty}} \frac{1}{(m-n)!} \sum_{\sigma\in S_n(m)[Z_2]} \rho_k(\alpha_k(\sigma)). \end{aligned}$$

But $\alpha_k(\sigma) = \beta_{2n,k}$ for $\sigma \in S_n(m)[Z_2]$ when $m \to \infty$, so we obtain $P_n|_{H_T^k} = \rho_k(\beta_{2n,k})$. $\diamondsuit$

APPENDIX F: Zorn's Lemma

We give a review of Zorn's Lemma following the presentation by Serge Lang [18] in his Analysis text.

Let S be a set. An *ordering* of S is a relation $\leq$ such that

1. $x \leq x$;
2. if $x \leq y$ and $y \leq z$, then $x \leq z$;
3. if $x \leq y$ and $y \leq x$, then $x = y$.

If any two elements of a set S are comparable relative to $\leq$, we have a *total order*.

Example 6.1. Let G be a group. Let S be the set of all subgroups of G. If H and H' are subgroups of G, define

$$H \leq H'$$

if H is a subgroup of H'. This relation is an order but not a total order.

Example 6.2. Let R be a ring; let S be the set of all left ideals of R with the order given by set inclusion.

Example 6.3. Let X be a set, and let S be the set of all subsets of X. Again, set inclusion gives an order on X.

Let S be an ordered set. By a *least element* of S (or *smallest element*), we mean an element $a \in S$ such that

$$a \leq x, \quad \forall x \in S.$$

We define similarly the *greatest element.*

By a *maximal element* $m \in S$, we mean an element such that

$$\text{if } x \in S \text{ and } m \leq x, \text{ then } x = m.$$

Note that there may be many maximal elements of S while greatest elements, if they exist, are unique.

Let S be an ordered set. We say that S is *totally ordered* if given $x, y \in S$, then we have either

$$x \leq y \quad \text{or} \quad y \leq x.$$

Let S be an ordered set, and T a subset. An *upper bound* of T in S is an element $b \in S$ such that

$$x \leq b, \quad \forall x \in T.$$

A *least upper bound* of T in S is an upper bound b of T in S such that, if c is another upper bound, then $b \leq c$.

We say S is *inductively ordered* if every non-empty totally ordered subset has an upper bound.

We say S is *strictly* inductively ordered if every non-empty totally ordered subset has a least upper bound.

Note that all three above examples are strictly inductively ordered.

Zorn's Lemma *Let S be a non-empty inductively ordered set. Then there exists a maximal element in S.*

APPENDIX G: Schur functions [9]

For a signature $\lambda = (\lambda_1, \cdots, \lambda_n)$, i.e. $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_n$, we define the rational function $A_\lambda(z) = A_\lambda(z_1, \cdots, z_n)$ on $\mathbb{C}^{*n}$ by

$$A_\lambda(z) = \begin{vmatrix} z_1^{\lambda_1} & z_1^{\lambda_2} & \cdots & z_1^{\lambda_n} \\ z_2^{\lambda_1} & z_2^{\lambda_2} & \cdots & z_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ z_n^{\lambda_1} & z_n^{\lambda_2} & \cdots & z_n^{\lambda_n} \end{vmatrix}.$$

In particular, for $\lambda = \delta := (n-1, \cdots, 1, 0)$, $A_\delta(z)$ is the Vandermonde polynomial

$$A_\delta(z) = V(z) := \prod_{1 \leq j < k \leq n} (z_j - z_k).$$

The Schur function s_λ is defined by

$$s_\lambda(z) = \frac{A_{\lambda+\delta}(z)}{V(z)}.$$

This is a symmetric rational function defined on $(\mathbb{C}^*)^n$, and Schur functions s_λ where $\lambda = (\lambda_1, \cdots, \lambda_n)$ run over all signatures of length $\leq n$, constitute a basis of the space of symmetric Laurent polynomials in n variables.

APPENDIX H: von Neumann algebras and their types [5] and [35]

Let $\mathcal{M}$ be an algebra over the the complex numbers. Let $x, y \in \mathcal{M},\ \lambda \in \mathbb{C}$. The algebra $\mathcal{M}$ is called a complex norm algebra if there is associated to each element x a real number $\|x\|$ satisfying properties:

1. $\|x\| \geq 0$ and $\|x\| = 0 \ \Leftrightarrow x = 0$,
2. $\|x + y\| \leq \|x\| + \|y\|$,
3. $\|\lambda x\| = |\lambda| \|x\|$,
4. $\|xy\| \leq \|x\| \|y\|$.

A Banach algebra is a complex normed algebra $\mathcal{M}$ which is complete with respect to the norm. An involution in $\mathcal{M}$ is a map $x \to x^*$ (called the adjoint of x) of $\mathcal{M}$ into itself such that

1. $(x^*)^* = x$
2. $(x + y)^* = x^* + y^*$,
3. $(\lambda x)^* = \overline{\lambda} x^*$,
4. $(xy)^* = y^* x^*$.

A Banach $*$-algebra is a complex Banach algebra $\mathcal{M}$ with an involution. It is sometimes called an involutive Banach algebra.

Definition 6.2. *A C^*-algebra is a Banach $*$-algebra such that $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{M}$.*

It follows easily from $\|x^*x\| = \|x^*\|\|x\|^2$ and from $x = (x^*)^*$ that $\|x^*\| = \|x\|$, thus an involution is an isometry. And a C^*-algebra need not have an identity operator. If the C^*-algebra has an identity operator, we call it unital C^*-algebra. For operators, we take next definition

Definition 6.3. *Let A be a Banach $*$-algebra. An element $x \in \mathcal{M}$ is*

1. *self-adjoint if $x = x^*$;*
2. *normal if $x^*x = xx^*$;*
3. *unitary if $x^*x = xx^* = 1$ provided $\mathcal{M}$ is unital;*
4. *projection if $x^2 = x$ and $x^* = x$;*

A subset M is called self-adjoint if $x \in M \leftrightarrow x^* \in M$.

Definition 6.4. *Let $\mathcal{M}$ be a C^*-algebra. An element x is called positive if $x = x^*$ and the spectrum of x^2 is contained in $[0, \infty]$. The set of positive elements of $\mathcal{M}$ denotes by $\mathcal{M}^+$.*

The examples of C^*-algebras are as follows:

Example 6.4. *Let H be a complex Hilbert space and $\mathcal{M} = B(H)$, the algebra of all bounded operators on H, is a C^*-algebra with the map $x \to x^*$.*

Example 6.5. *A norm-closed self-adjoint subalgebra of $B(H)$ is a C^*-algebra and it is called a concrete C^*-algebra.*

Example 6.6. *The map $x \to \overline{x}$ (where $\overline{x}$ is the complex conjugate of x on $\mathcal{M} = \mathbb{C}$ is an involution with which $\mathcal{M}$ is a a commutative C^*-algebra.*

[2] The spectrum of x is the set $\sigma(x) := \{\lambda \in \mathbb{C} : \lambda I - x$ is not invertible $\}$.

Example 6.7. *Let X be a locally compact separable space, and $\mathcal{M} = C_0(X)$, the algebra of complex-valued continuous function vanishing at infinity on X, is a commutative C^*-algebra with the map $f \to \overline{f}$.*

We will define a W^*-algebra (sometimes called a von Neumann algebra) which is very important in the representation theory of Banach $*$-algebras.

Definition 6.5. *A W^*-algebra $\mathcal{M}$ is a C^* subalgebra of $L(H)$ containing the identity operator in the Hilbert space H which is closed in the weak operator topology.*

For each subset M of $L(H)$, we define M' as the commutant of M:

$$M' = \{x \in L(H) : xy = yx \text{ for all } y \in M\}.$$

If M is a self-adjoint then M' is a self-adjoint unital algebra so it is a C^*-algebra. Moreover, M' is a weakly closed algebra and the commutant of a C^*-algebra is always a von Neumann algebra. The double and triple commutant of M can be expressed as $(M')'$ and $((M')'))'$ respectively. We will state the double commutant theorem.

Theorem 6.2. *Let M be a C^*-subalgebra of $L(H)$ containing the identity operator on H. The following conditions are equivalent:*

1. *$M = M''$,*
2. *M is closed in the weak operator topology,*
3. *M is closed in the strong operator topology.*

Definition 6.6. *The center of a W^* -algebra $\mathcal{M}$ is*

$$Z(\mathcal{M}) = \{x \in \mathcal{M} : xy = yx, y \in \mathcal{M}\}.$$

Definition 6.7. *A W^*-algebra $\mathcal{M}$ is called a factor if its center consists of scalar multiples of identity. In other words, a factor is a W^*-algebra $\mathcal{M}$ with trivial center.*

Let $\mathcal{M}$ will denote a W^*-algebra. An element $x \in \mathcal{M}$ is a projection if $x^2 = x$ and $x^* = x$. A projection x is called finite if there is no projection $y < x$ that is equivalent to x.

Definition 6.8. *Let $\mathcal{M}^+$ be the set of positive elements of W^*-algebra $\mathcal{M}$. A trace on $\mathcal{M}^+$ is a function $\tau : \mathcal{M}^+ \to [0, \infty]$ satisfying the following*

1. $\tau(x + y) = \tau(x) + \tau(y)$ *for all* $x,\ y \in \mathcal{M}^+$,
2. $\tau(\lambda x) = \lambda\tau(x)$ *for all* $x \in \mathcal{M}^+$ *and* λ *is nonnegative real number,*
3. *If* $x \in \mathcal{M}^+$ *and* u *is unitary in* $\mathcal{M}$, $\tau(u^* x u) = \tau(x)$.

A trace τ is called

1. *faithful if* $\tau(x) = 0 \Rightarrow x = 0$,
2. *finite if* $\tau(x) < \infty$ *for all* $x \in \mathcal{M}^+$,
3. *semifinite if for all nonzero* $x \in \mathcal{M}^+$, *there exists a non-zero element* y *in* $\mathcal{M}^+$ *with* $\tau(y) < \infty$ *and* $y \leq x$.

We classify factors in the following way:

Definition 6.9. *A factor is of*

1. *type I if the set of traces of all the projections is discrete. All the type I factors on a separable Hilbert space are either I_n $(trace = 0)$ or I_∞ $(trace = \infty)$.*
2. *type II_1 if the set of traces of projections is* $[0, 1]$,
3. *type II_∞ if the set of traces of projections is* $[0, \infty]$,
4. *type III if it does not contain any nonzero finite projections.*

W^*-algebra is of type I (resp, II, or III) if every factor of it is of type I (resp, II, or III).

VITA

Yun Yoo was born in Seoul, South Korea. In 1997 he moved to the United States and he attended Drexel University to realize his dream to become a mathematics professor. He graduated with a 4.0 G.P.A. from Drexel University with a combined Bachelor's and Master's Degree in Mathematics in 2002. During 2002 to 2004, he has been teaching Mathematics at University of Pennsylvania, Community College of Philadelphia and Temple University as a part-time faculty. During his time for teaching assistant at Drexel University between 2007 and 2010, he has been a calculus team member and he received Teaching Excellence Award for Graduate Students in 2009. He is currently a full-time faculty at Community College of Philadelphia. He became a tenured Assistant Professor in 2010.